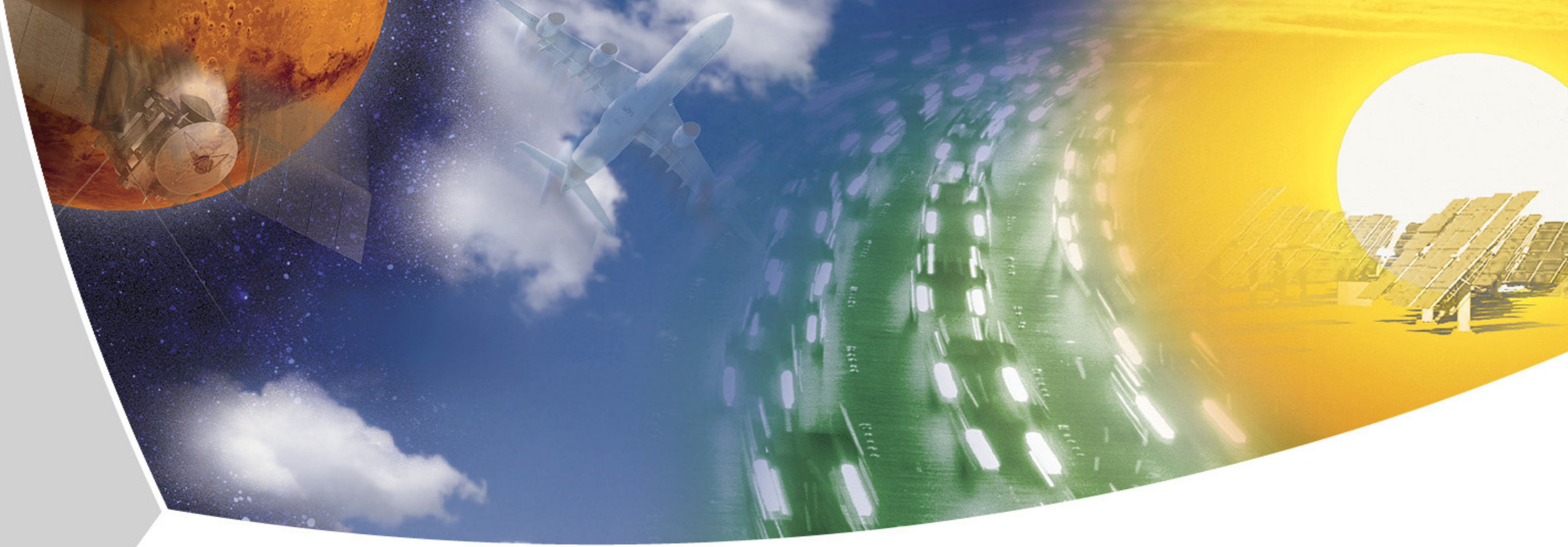


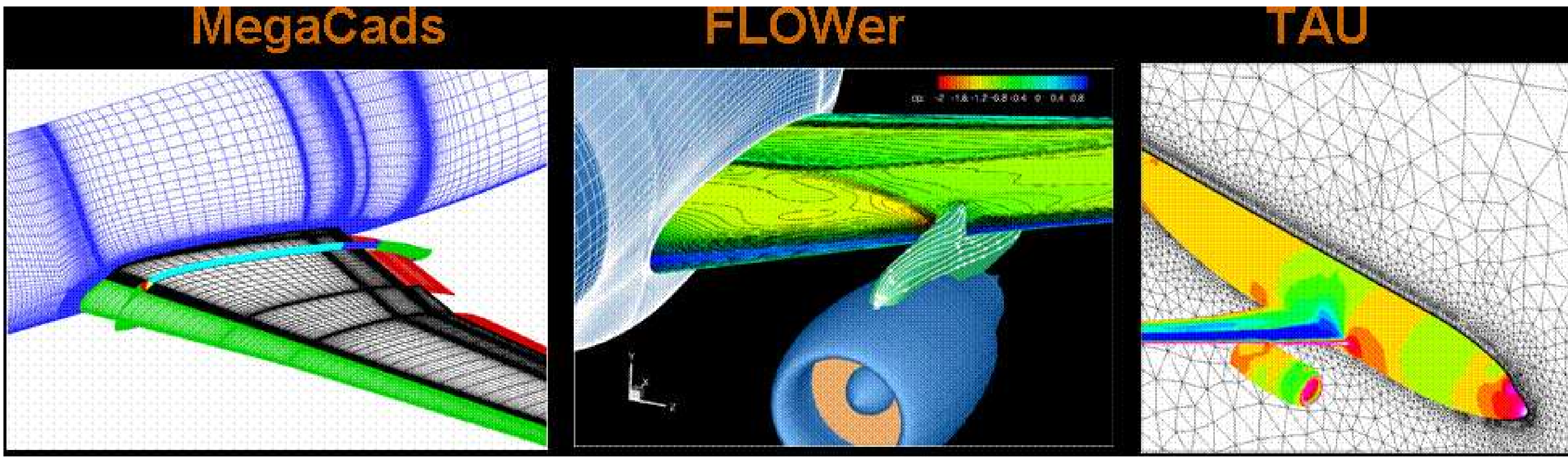
Discontinuous Galerkin methods for compressible flows: higher order accuracy, error estimation and adaptivity

Ralf Hartmann

DLR Deutsches Zentrum für Luft- und Raumfahrt e.V.
in der Helmholtz-Gemeinschaft

DLR, Institute of Aerodynamics and Flow Technology

- **structured grid generator: MegaCads**
- **structured Finite Volume flow solver: Flower**
- **unstructured Finite Volume flow solver: TAU**



used in industry: Airbus, EADS-M, ...

Motivation

Higher order methods:

- ▶ Numerical resolution and tracking of vortices
  - Helicopters: Vortex creation and blade-vortex interaction
  - Transport aircrafts: wake-vortices
- ▶ Numerical resolution of viscous boundary layers
- ▶ Numerical approximation of aerodynamical forces: lift, drag, moments

Error estimation:

- ▶ Reliable prediction of aerodynamical forces

Adaptivity:

- ▶ Mesh refinement for better resolution of vortices, boundary layers, etc.
- ▶ Goal-oriented mesh refinement for accurate approximation of aerodynamical forces

Overview

Part I: DG discretization of the compressible Euler and Navier-Stokes equations

- Solution method (fully implicit)
- Higher order computational results
- A posteriori error estimation
- Isotropic and anisotropic adjoint-based (goal-oriented) adaptivity

Part II: Adjoint consistent Discontinuous Galerkin discretizations

- Definition of adjoint consistency for linear and nonlinear problems
- Consistent modification of target functionals
- Adjoint consistent discretizations
  - SIPG for Poisson's equation
  - DG discretization of the compressible Euler equations
  - SIPG discretization of the compressible Navier-Stokes equations
- Numerical examples

Publications: Part I

- R. Hartmann and P. Houston
  Symmetric interior penalty DG methods for the compressible Navier-Stokes equations I: Method formulation
  Int. J. Numer. Anal. Model. 3(1): 1-20, 2006.

- R. Hartmann and P. Houston
  Symmetric interior penalty DG methods for the compressible Navier-Stokes equations II: Goal–oriented a posteriori error estimation
  Int. J. Numer. Anal. Model. 3(2): 141-162, 2006.

- R. Hartmann
  Adaptive discontinuous Galerkin methods with shock-capturing for the compressible Navier-Stokes equations
  Int. J. Numer. Meth. Fluids 51(9-10): 1131–1156, 2006.

- T. Leicht
  Anisotropic mesh refinement for Discontinuous Galerkin methods in aerodynamic flow simulations
  Master Thesis, DLR & TU Dresden.

Publications: Part II

- R. Hartmann
  Derivation of an adjoint consistent discontinuous Galerkin discretization of the compressible Euler equations
  In Gert Lube and Gerd Rapin (eds.),
  Proceedings of the BAIL 2006 conference. To appear.

- R. Hartmann
  Adjoint consistency analysis of Discontinuous Galerkin discretizations
  SIAM J. Numer. Anal. (2006). Submitted.

- R. Hartmann et al.
  PADGE: The DG flow solver of the DLR.

- W. Bangerth, R. Hartmann and G. Kanschat
  `deal.II` C++ FEM library, http://www.dealii.org

The compr. Euler and Navier-Stokes equations in 2D

The compressible Euler equations:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varrho \\ \varrho v_1 \\ \varrho v_2 \\ \varrho E \end{pmatrix} + \frac{\partial}{\partial x_1}\begin{pmatrix} \varrho v_1 \\ \varrho v_1^2 + p \\ \varrho v_1 v_2 \\ v_1(\varrho E + p) \end{pmatrix} + \frac{\partial}{\partial x_2}\begin{pmatrix} \varrho v_2 \\ \varrho v_1 v_2 \\ \varrho v_2^2 + p \\ v_2(\varrho E + p) \end{pmatrix} = 0$$

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$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0$$

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The compressible Navier-Stokes equations:

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathcal{F}^c(\mathbf{u}) - \nabla \cdot \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = 0$$

The compr. Euler and Navier-Stokes equations in 2D

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$$\frac{\partial}{\partial t}\begin{pmatrix}\varrho\\ \varrho v_1\\ \varrho v_2\\ \varrho E\end{pmatrix}+\frac{\partial}{\partial x_1}\begin{pmatrix}\varrho v_1\\ \varrho v_1^2+p\\ \varrho v_1 v_2\\ v_1(\varrho E+p)\end{pmatrix}+\frac{\partial}{\partial x_2}\begin{pmatrix}\varrho v_2\\ \varrho v_1 v_2\\ \varrho v_2^2+p\\ v_2(\varrho E+p)\end{pmatrix}=0$$

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The compressible Navier-Stokes equations:

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$$\mathbf{f}_1^v(\mathbf{u},\nabla\mathbf{u})=\begin{pmatrix}0\\ \tau_{11}\\ \tau_{21}\\ \tau_{11}v_1+\tau_{12}v_2+\kappa T_{x_1}\end{pmatrix},\qquad \mathbf{f}_2^v(\mathbf{u},\nabla\mathbf{u})=\begin{pmatrix}0\\ \tau_{12}\\ \tau_{22}\\ \tau_{21}v_1+\tau_{22}v_2+\kappa T_{x_2}\end{pmatrix}.$$

DG discretization of the compr. Euler equations

The problem:

$$\nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0 \quad \textbf{in } \Omega,$$

with $\mathbf{u} = (\varrho, \varrho v_1, \varrho v_2, \rho E)^T$**.**

The discretization of DG(p): Find u_h in $\mathrm{V}_{h,p}$ such that

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) \equiv \sum_{\kappa \in \mathcal{T}_h} \left\{ - \int_\kappa \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \int_{\partial\kappa \setminus \Gamma} \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}_\kappa)\, \mathbf{v}_h^+ \,\mathrm{d}s \right\}$$
$$+ \int_\Gamma \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}_\kappa)\, \mathbf{v}_h^+ \,\mathrm{d}s = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_{h,p},$$

with

$$\mathbf{V}_{h,p} = \left\{ \mathbf{v} \in [L_2(\Omega)]^4 : \mathbf{v}|_\kappa \in [\mathcal{Q}_p(\kappa)]^4 \;\; \forall \kappa \in \mathcal{T}_h \right\},$$

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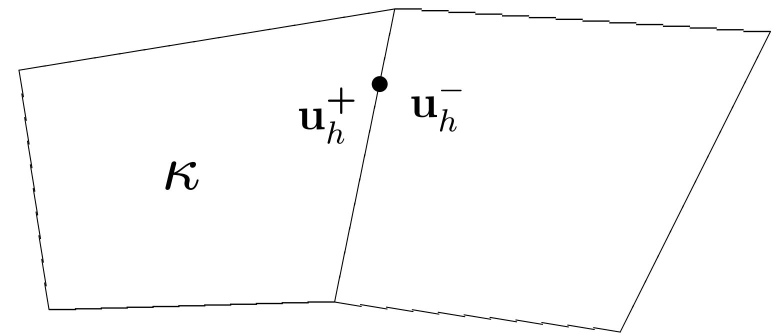

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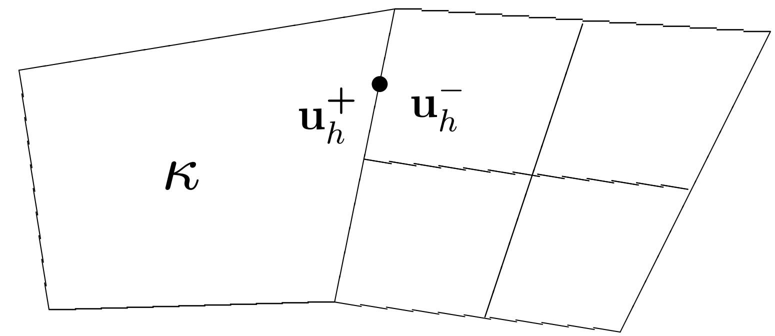

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DG discretization of the compr. Navier-Stokes equations

Find $\mathbf{u}_h$ in $\mathrm{V}_{h,p}$ such that

$$\begin{aligned}\mathcal{N}(\mathbf{u}_h,\mathbf{v}_h) \equiv &-\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\backslash\Gamma} \mathcal{H}(\mathbf{u}_h^+,\mathbf{u}_h^-,\mathbf{n}_\kappa)\cdot\mathbf{v}_h^+ \,\mathrm{d}s \\ &+\int_\Omega \mathcal{F}^v(\mathbf{u}_h,\nabla_h\mathbf{u}_h) : \nabla_h\mathbf{v}_h\,\mathrm{d}\mathbf{x} - \int_{\Gamma_\mathcal{I}} \{\!\{G(\mathbf{u}_h)\nabla\mathbf{u}_h\}\!\} : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s \\ &-\int_{\Gamma_\mathcal{I}} \{\!\{G^\top(\mathbf{u}_h)\nabla\mathbf{v}_h\}\!\} : \underline{[\![\mathbf{u}_h]\!]})\,\mathrm{d}s + \int_{\Gamma_\mathcal{I}} \delta\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s + \mathcal{N}_\Gamma(\mathbf{u}_h,\mathbf{v}_h) = 0\end{aligned}$$

for all $\mathbf{v}_h$ in $\mathrm{V}_{h,p}$, with

$$\begin{aligned}\mathcal{N}_\Gamma(\mathbf{u}_h,\mathbf{v}_h) \equiv &\int_\Gamma \mathcal{H}_\Gamma(\mathbf{u}_h^+,\mathbf{u}_\Gamma(\mathbf{u}_h^+),\mathbf{n})\cdot\mathbf{v}_h^+\,\mathrm{d}s + \int_\Gamma \delta\left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\cdot\mathbf{v}_h^+\,\mathrm{d}s, \\ &-\int_\Gamma G_\Gamma(\mathbf{u}_h)\nabla\mathbf{u}_h : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s \\ &-\int_\Gamma G_\Gamma^\top(\mathbf{u}_h)\nabla\mathbf{v}_h : \left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\otimes\mathbf{n}\,\mathrm{d}s\end{aligned}$$

Newton iteration

DG discretization: Find u_h in $\mathrm{S}_{h,p}$ such that

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{S}_{h,p}.$$

Newton iteration: Given u_h^n, find d_h^n in $\mathrm{S}_{h,p}$ such that

$$\mathcal{N}'_{\mathbf{u}}[\mathbf{u}_h^n](\mathbf{d}_h^n, \mathbf{v}_h) = \mathcal{R}(\mathbf{u}_h^n, \mathbf{v}_h) \equiv -\mathcal{N}(\mathbf{u}_h^n, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{S}_{h,p},$$

and compute update

$$\mathbf{u}_h^{n+1} = \mathbf{u}_h^n + \omega^n \mathbf{d}_h^n.$$

Laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ flow around the NACA0012 airfoil

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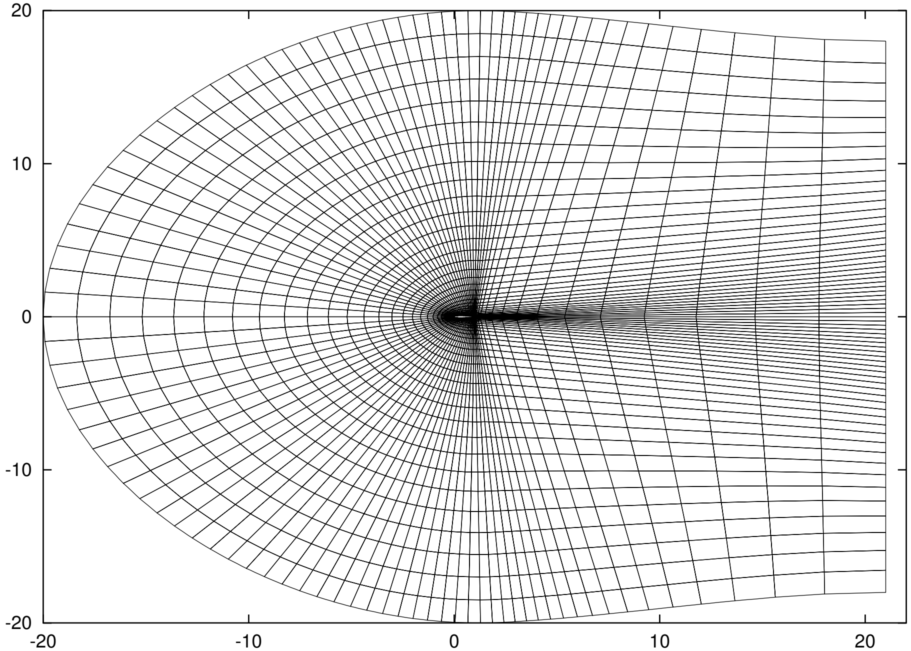
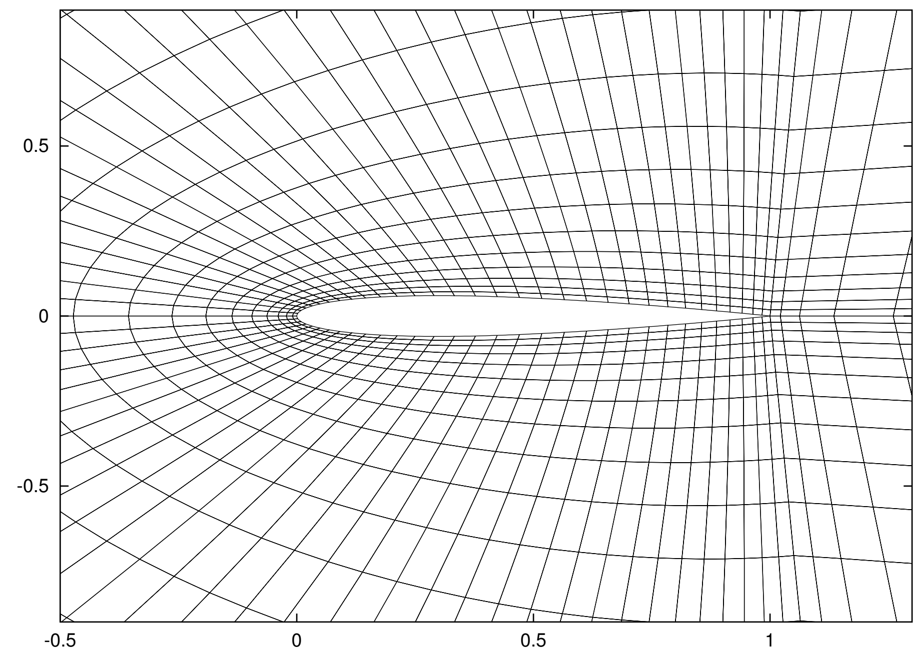
Grid of 3072 cells:



Zoom of this grid:



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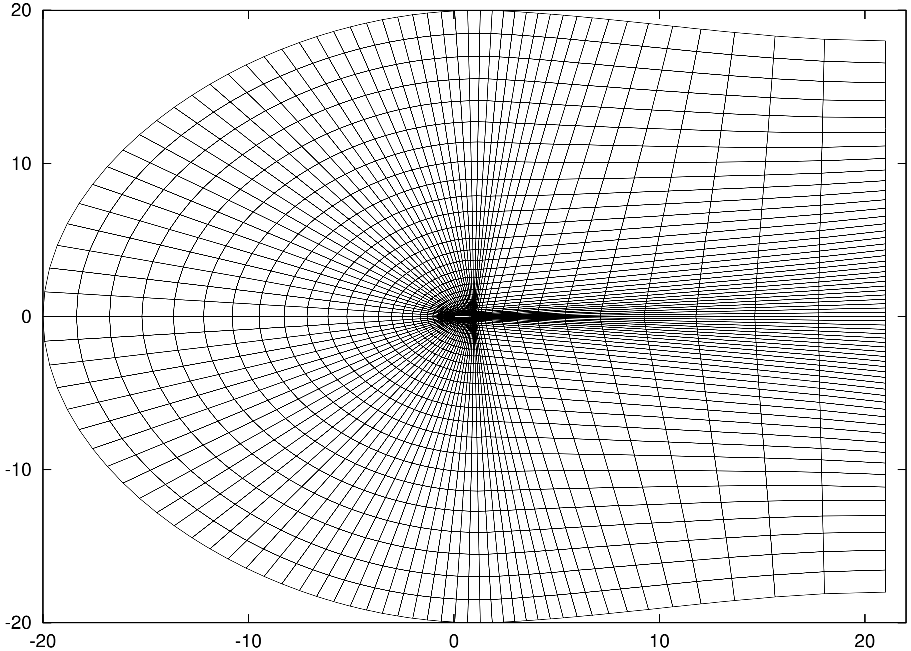

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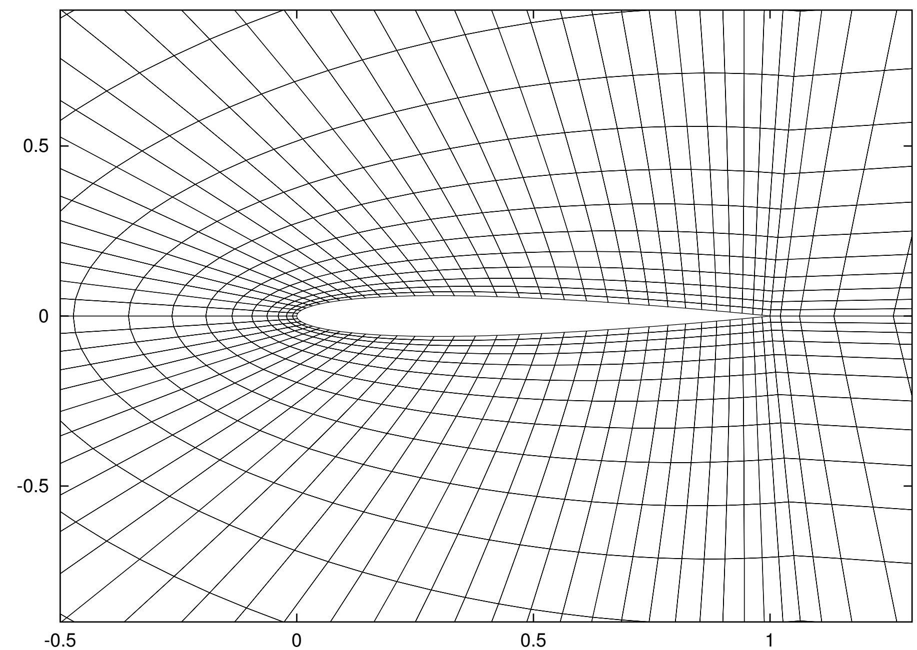

Computation using DG(p), $p = 1, 2, 3$ on sequence of globally refined grids of 3072, 12288, 49152 and 196608 cells

Fully implicit solver: Newton iteration

Each Newton step is solved using GMRES and an ILU preconditioner

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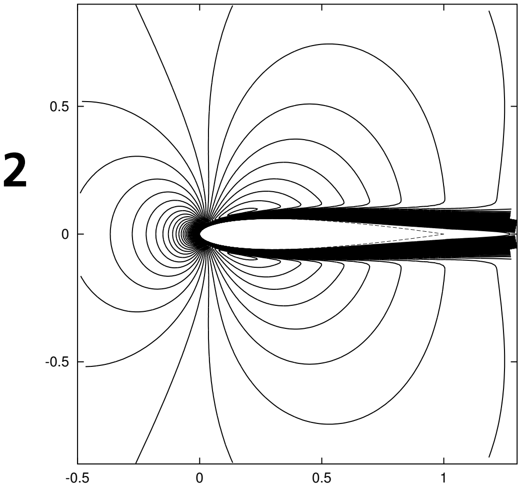

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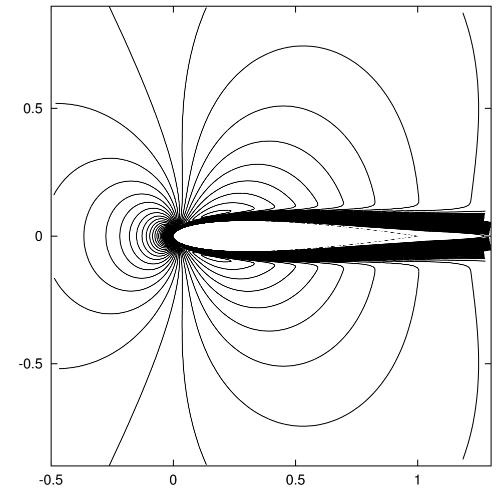
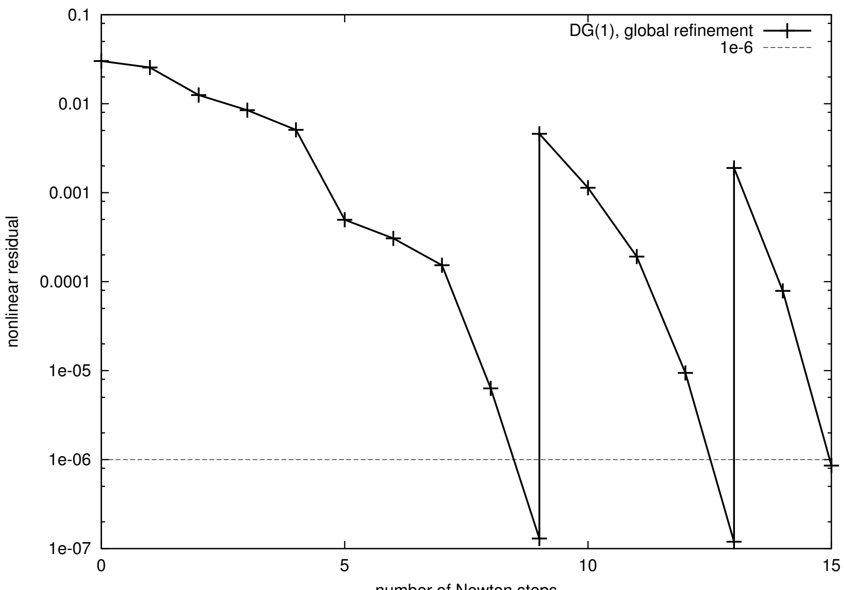


$M = 0.5$, $Re = 5000$, $\alpha = 0$ **flow around NACA0012**

non-linear residuals (DG(1), global refinement)



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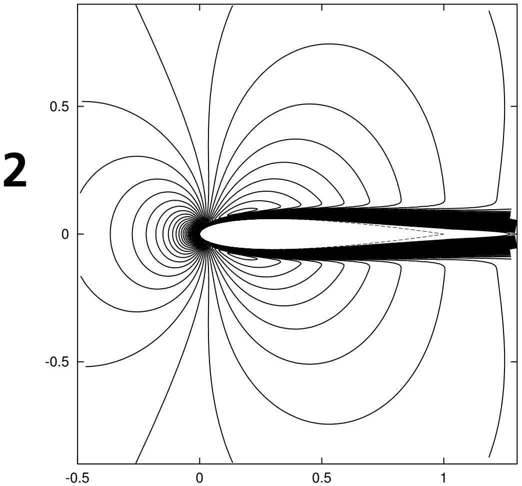

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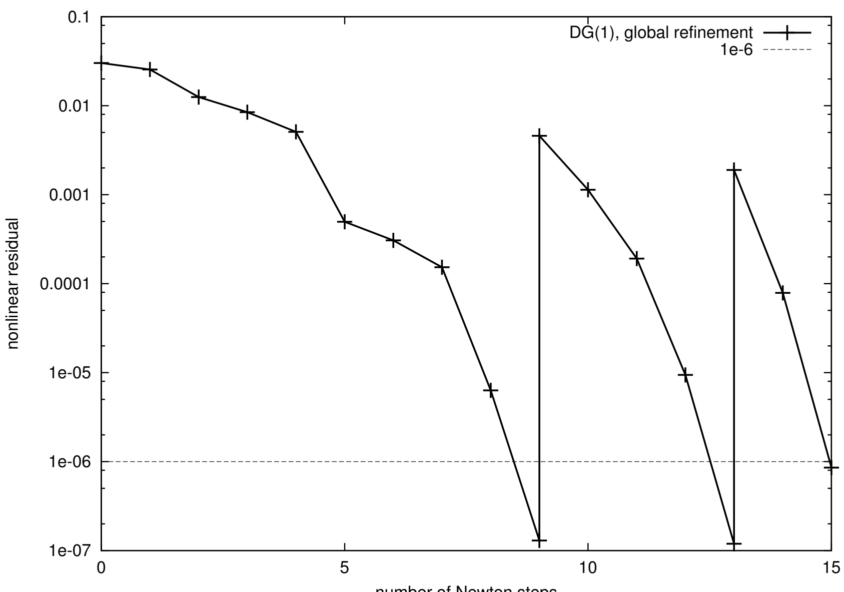

mesh	DG(1)	DG(2)	DG(3)
1	9	$4^\star$	$4^\star$
2	4	2	2
3	2	2	

* : on a DG(1) pre-iterated solution

Higher order computations for laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ **flow around the NACA0012 airfoil**

Convergence of cdp and cdf under global refinement

cdp (pressure induced drag)

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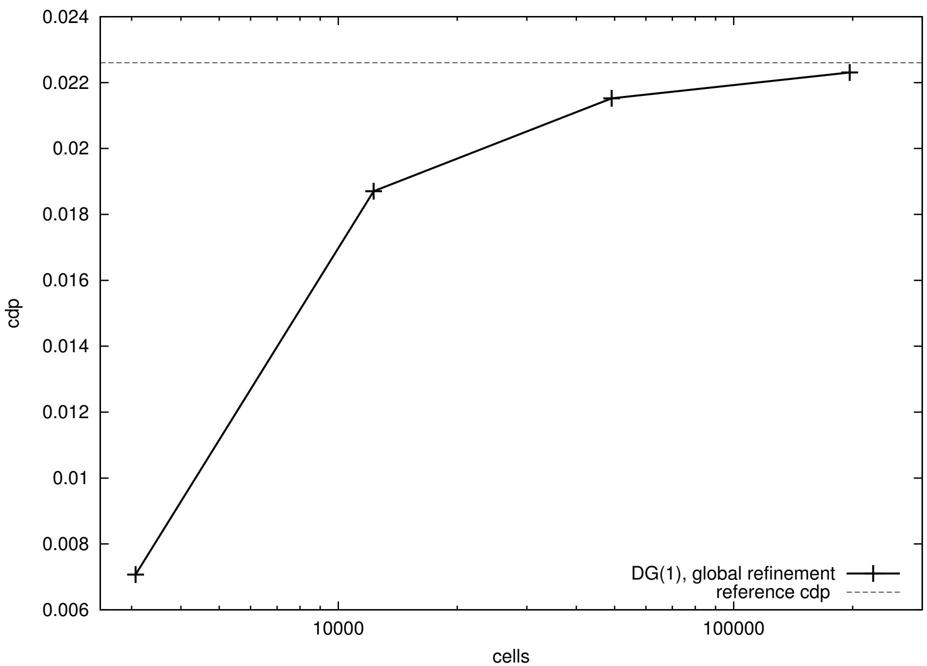

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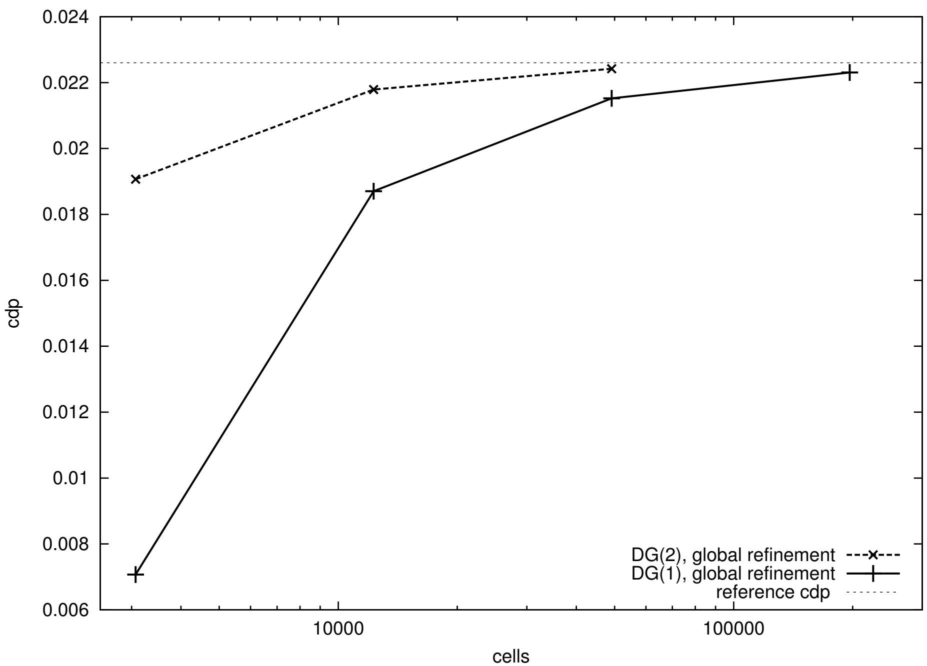

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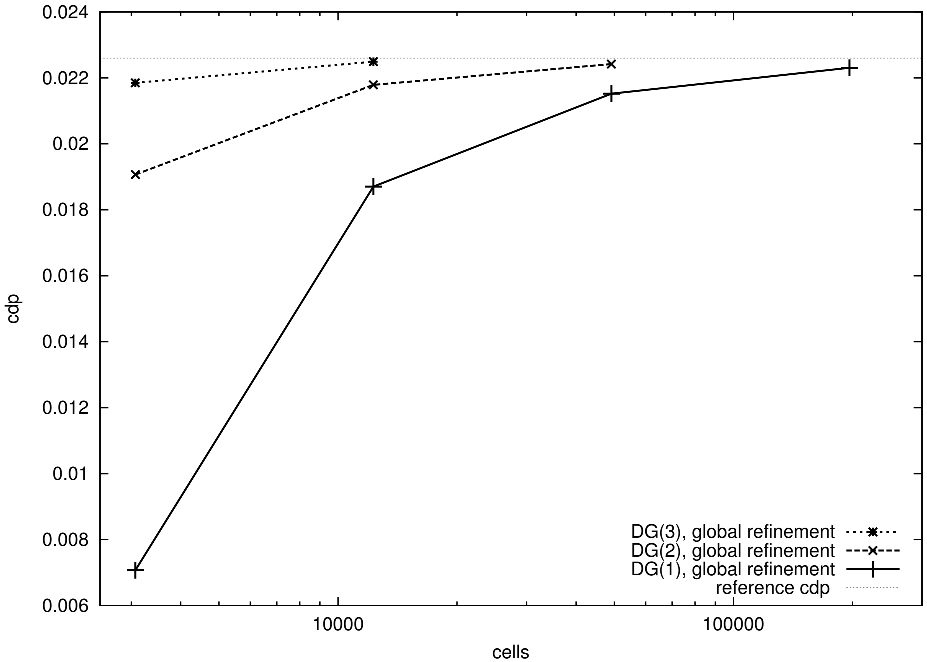

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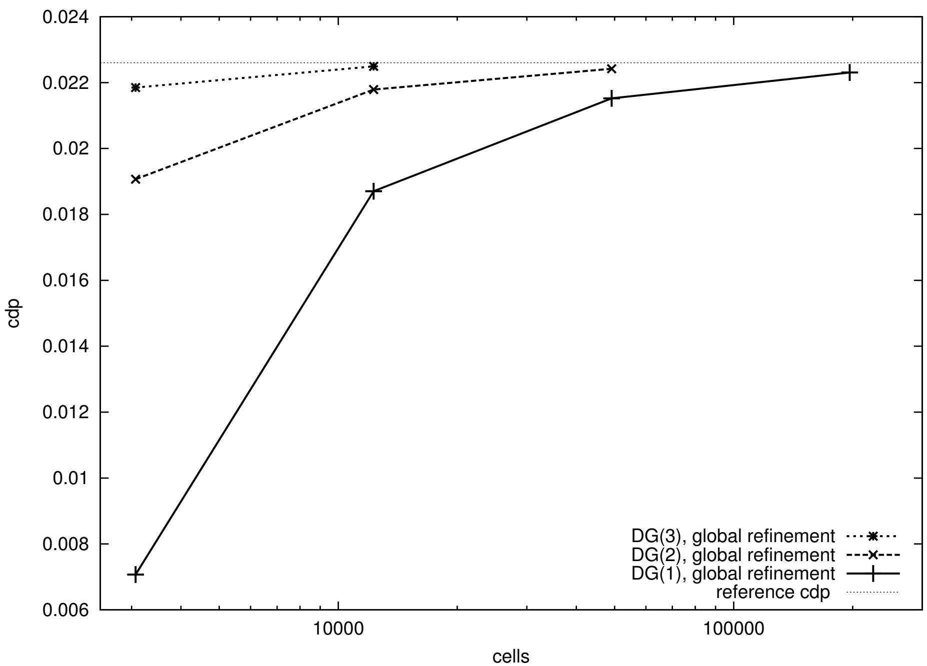
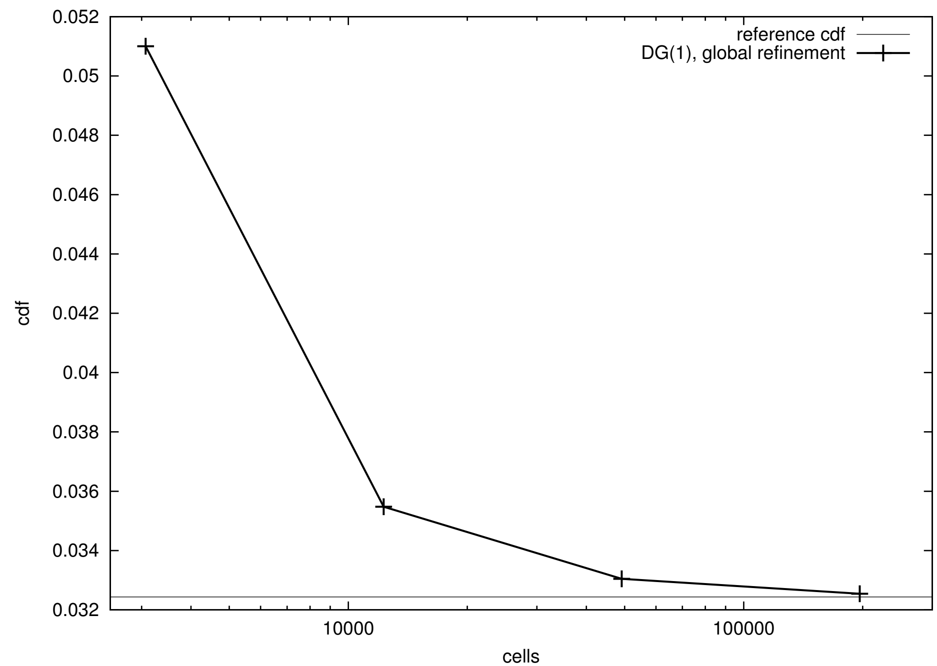


cdf (viscous drag)



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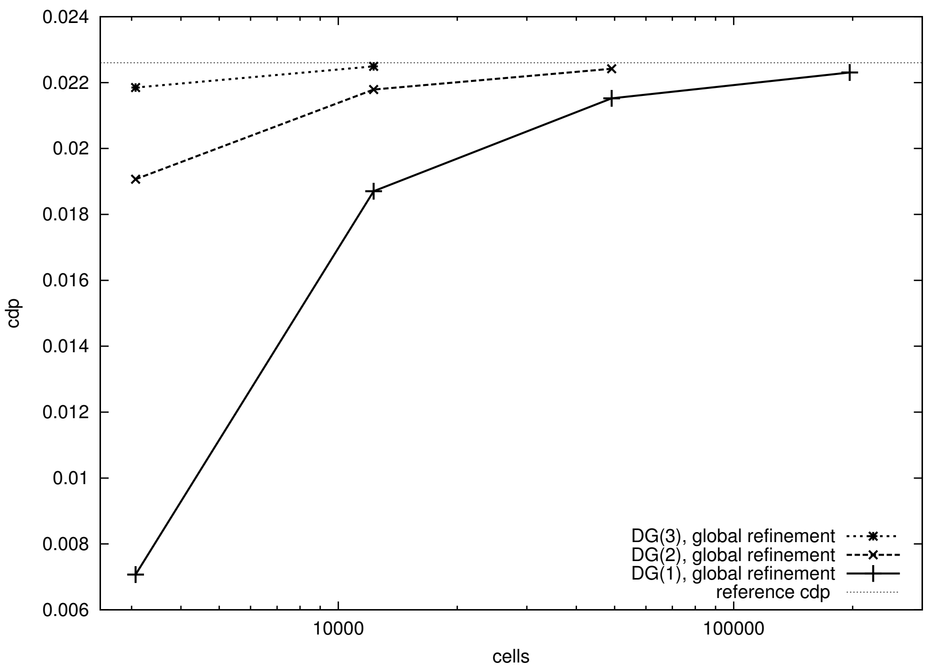

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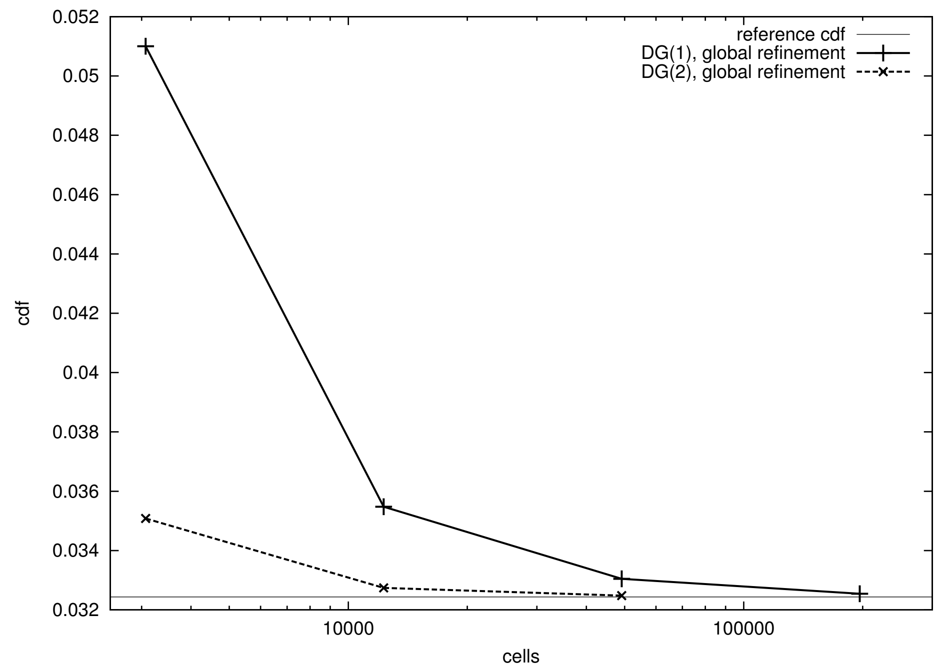

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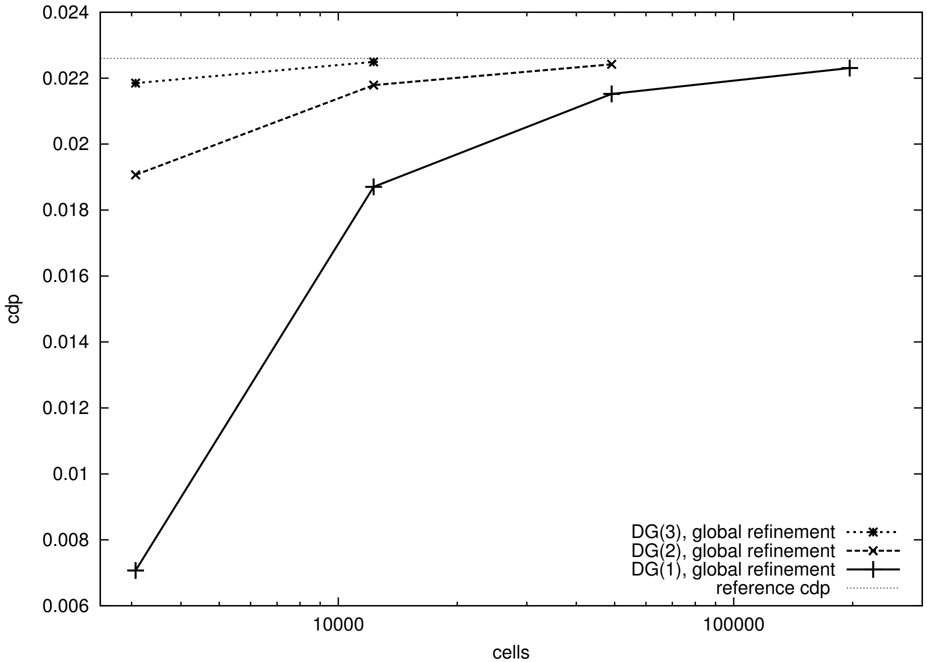

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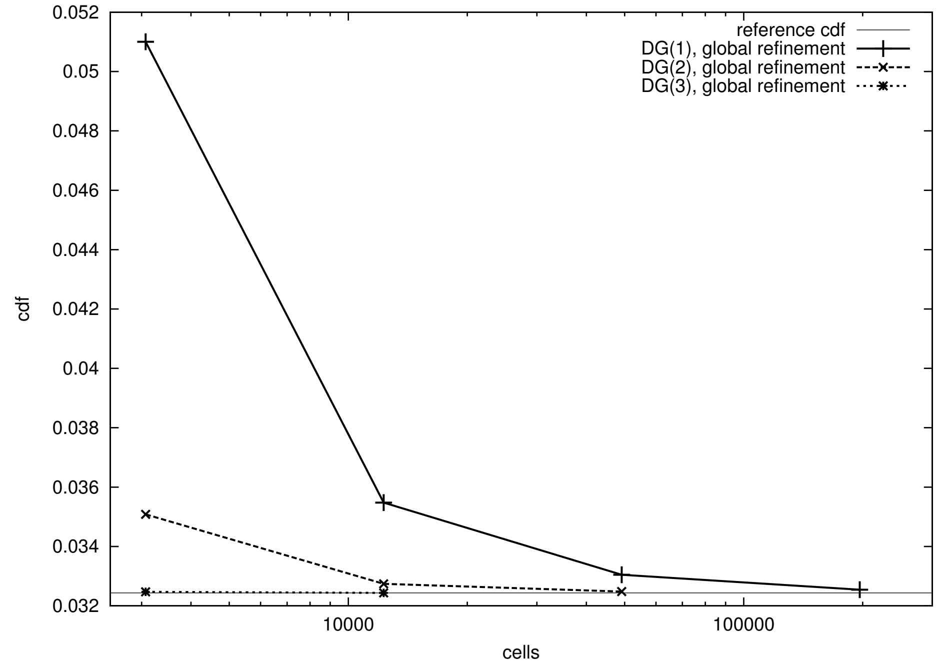
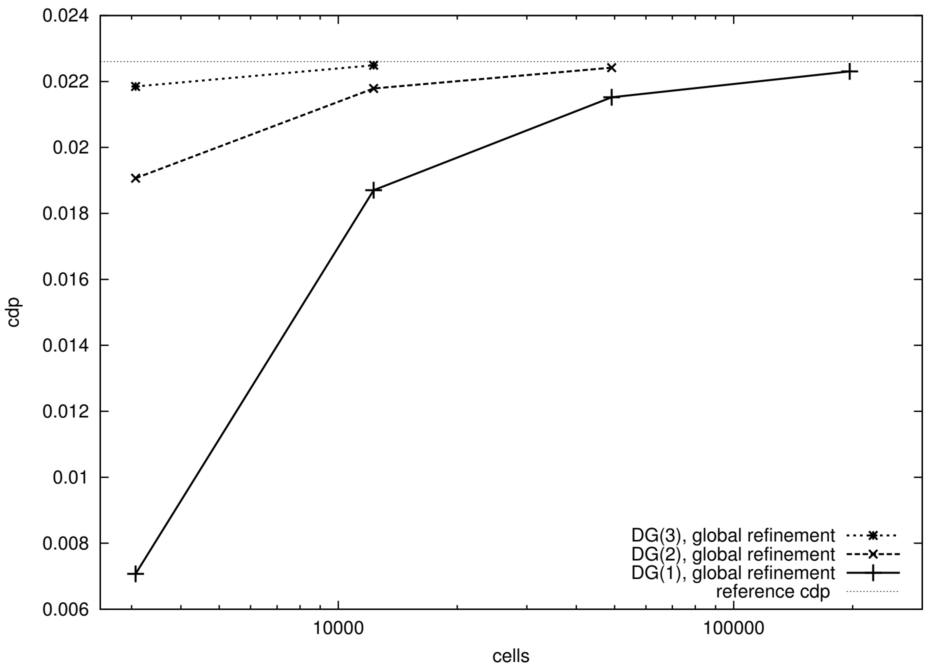


Convergence of cdp

cdp



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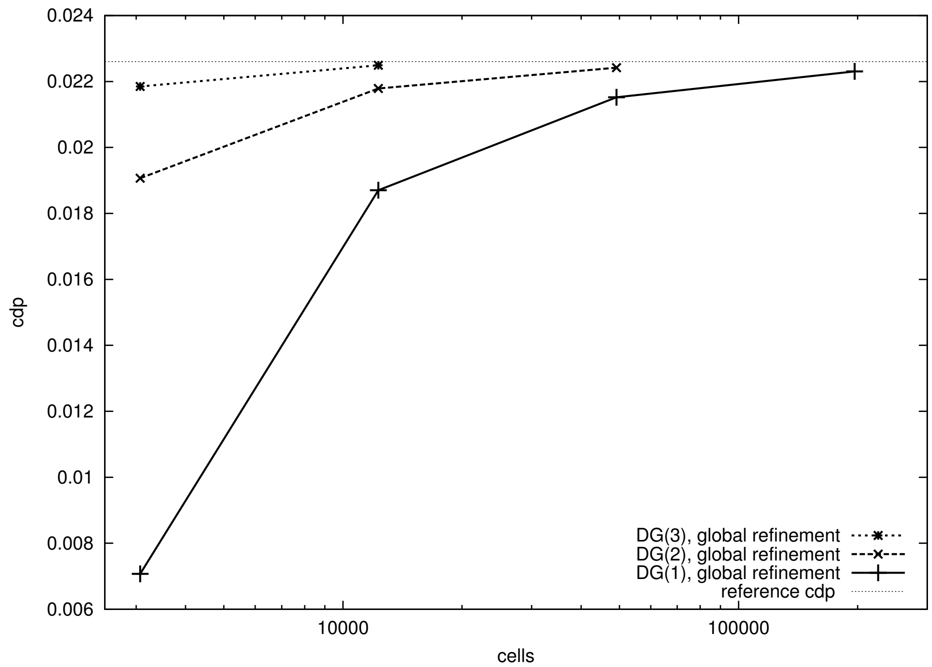
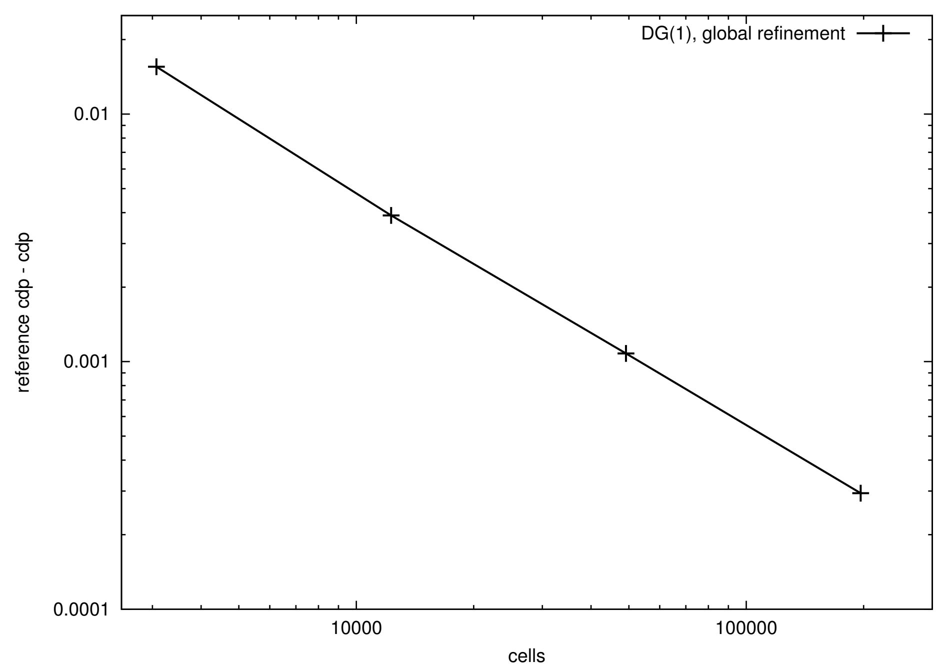


reference cdp - cdp



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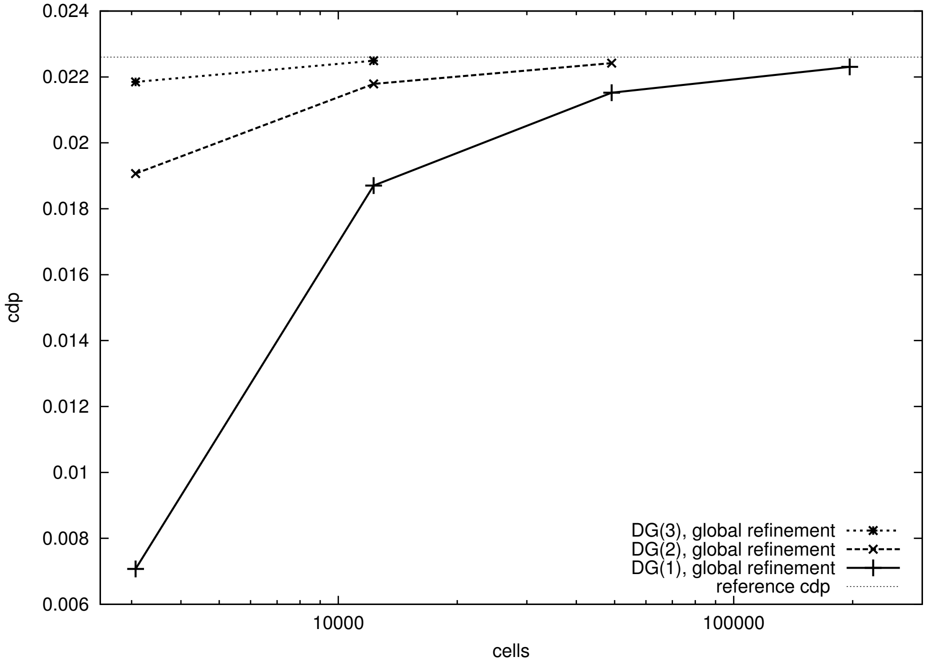

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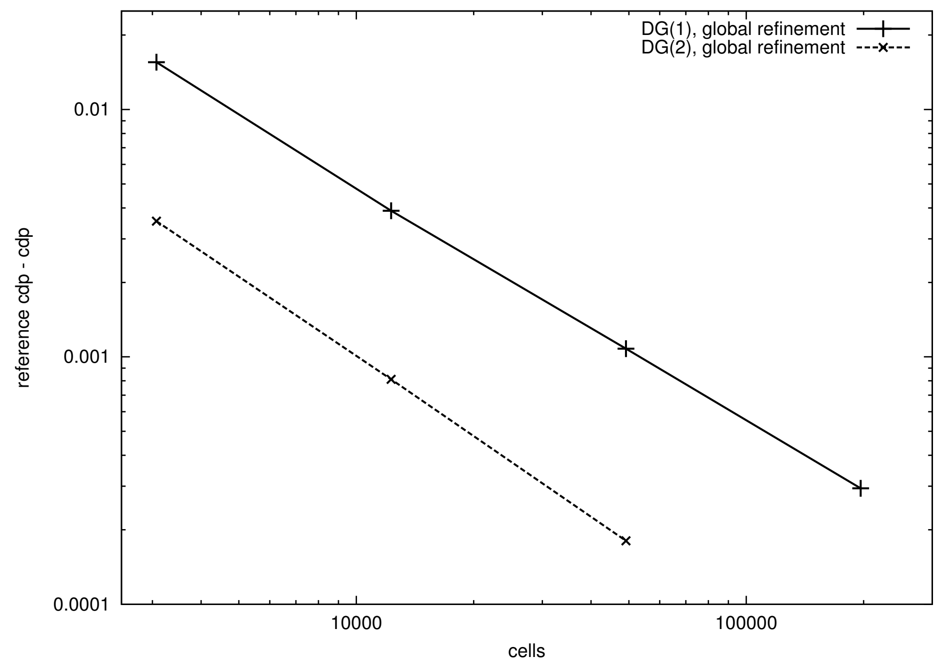

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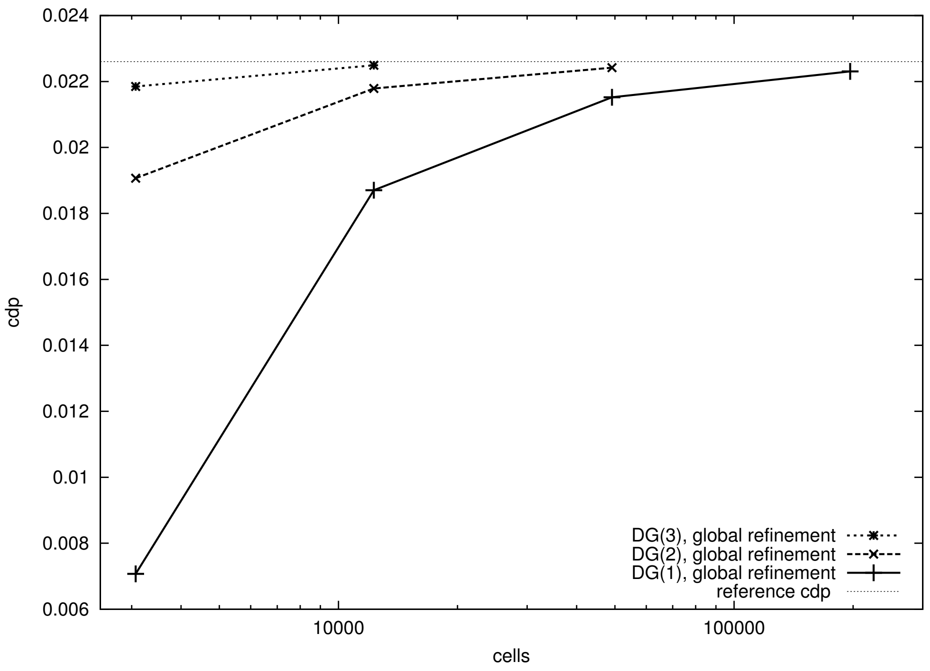

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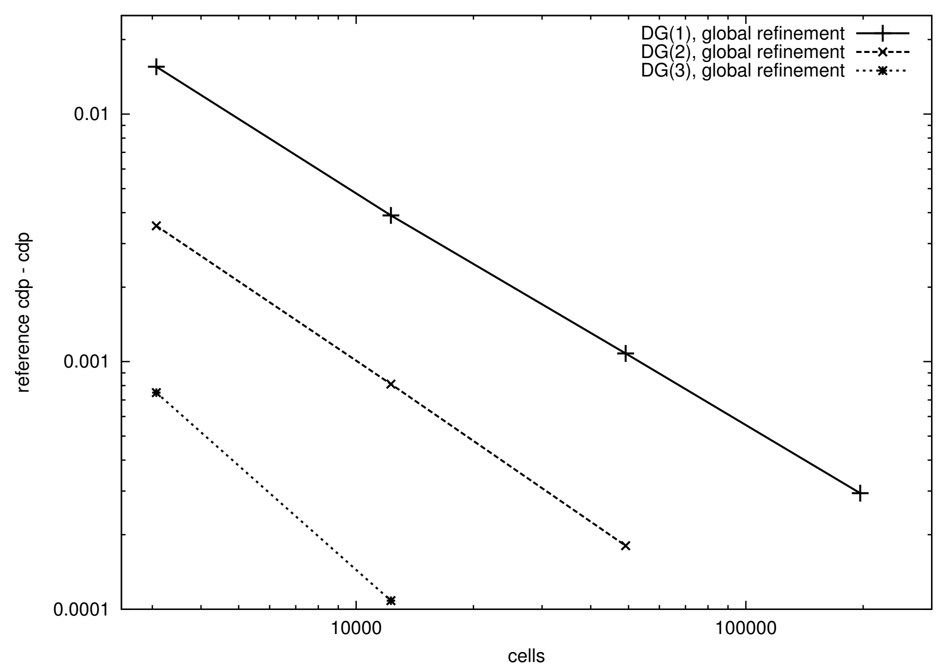

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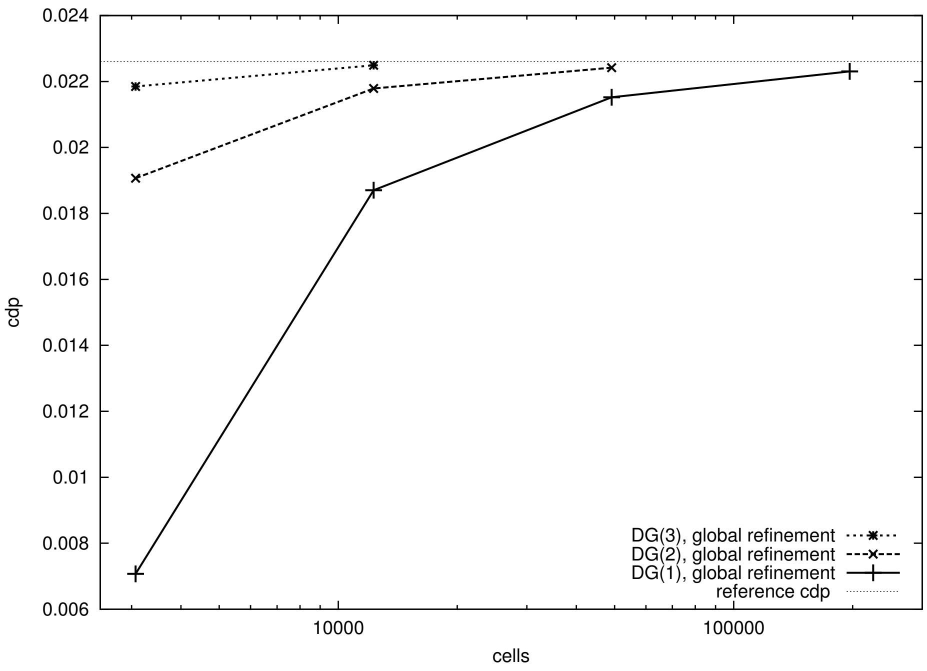

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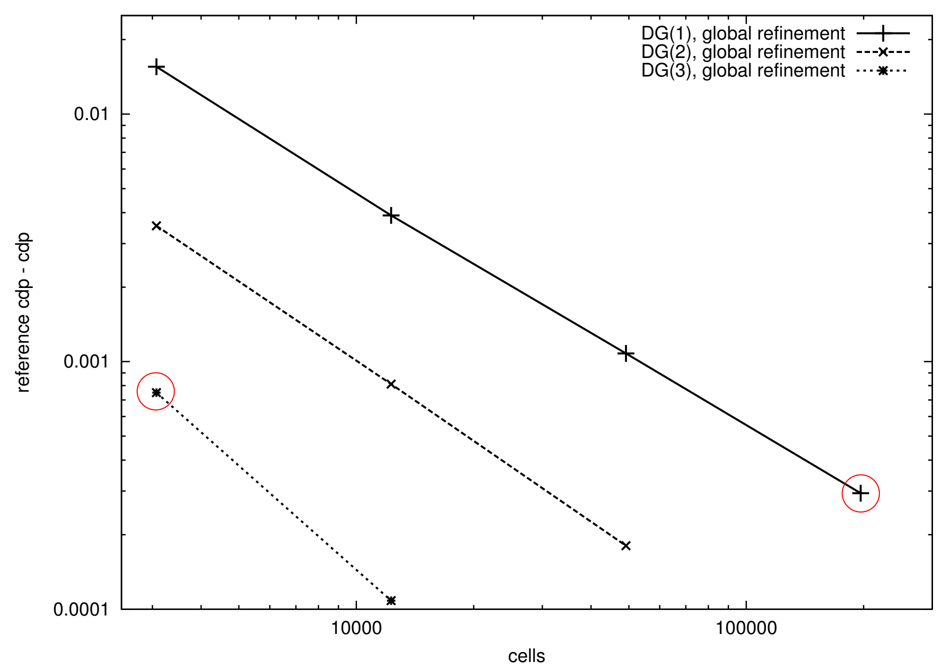
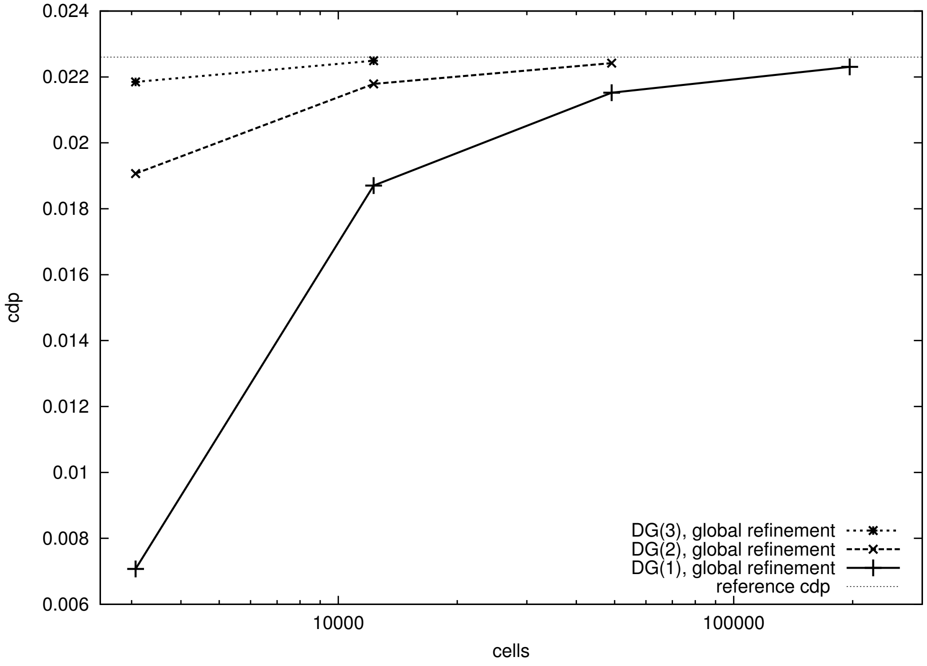
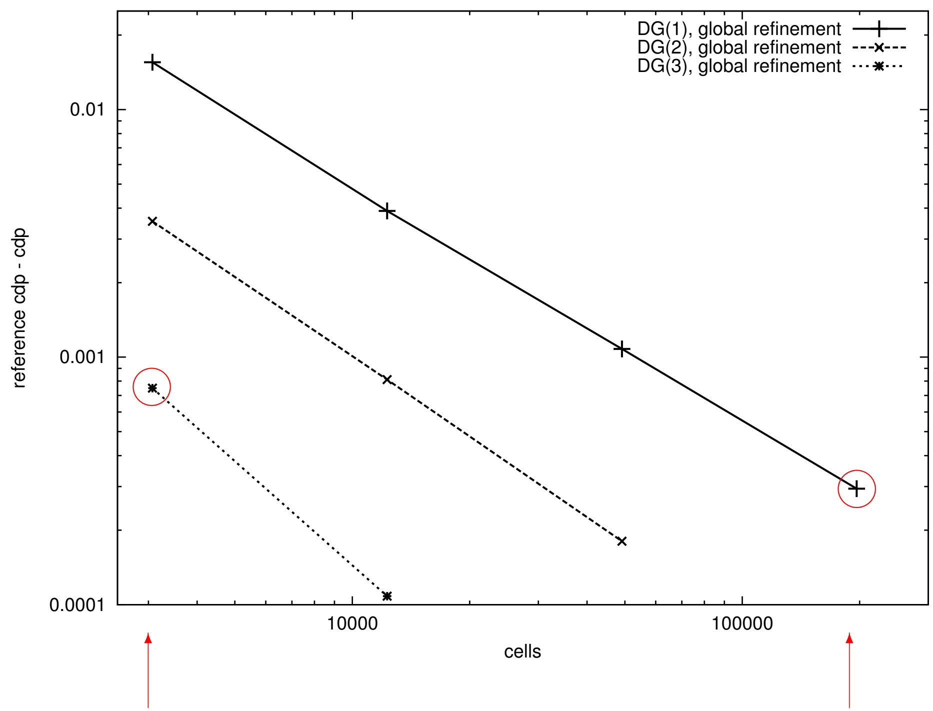


Convergence of cdp

cdp

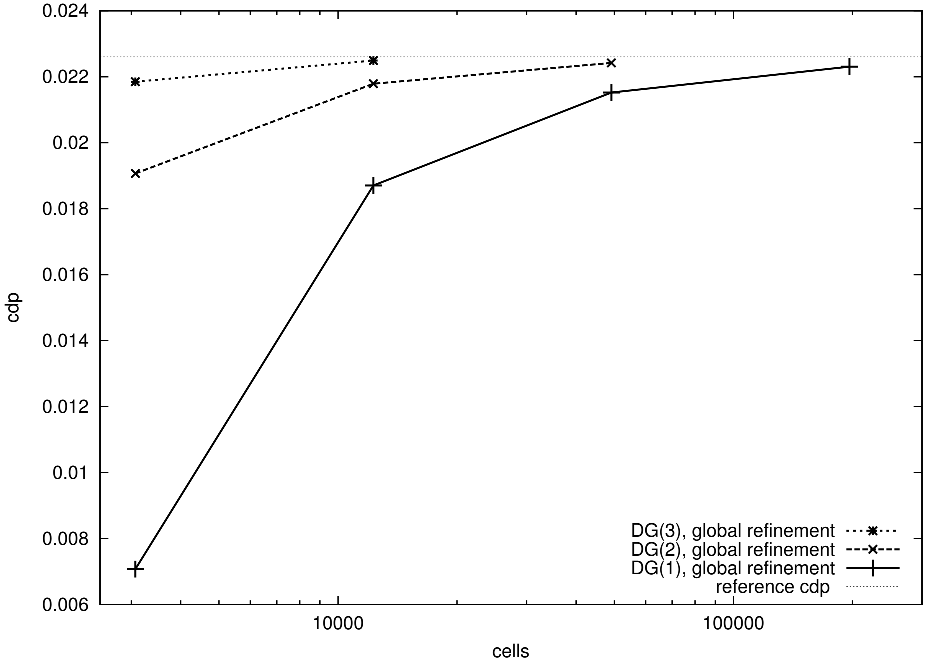


reference cdp - cdp



Convergence of cdp

cdp



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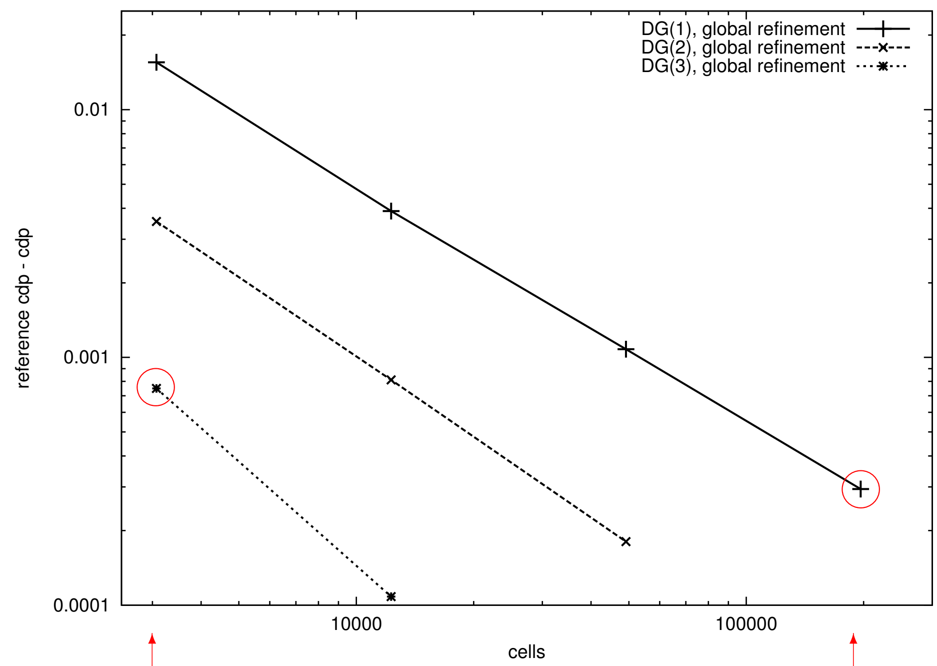
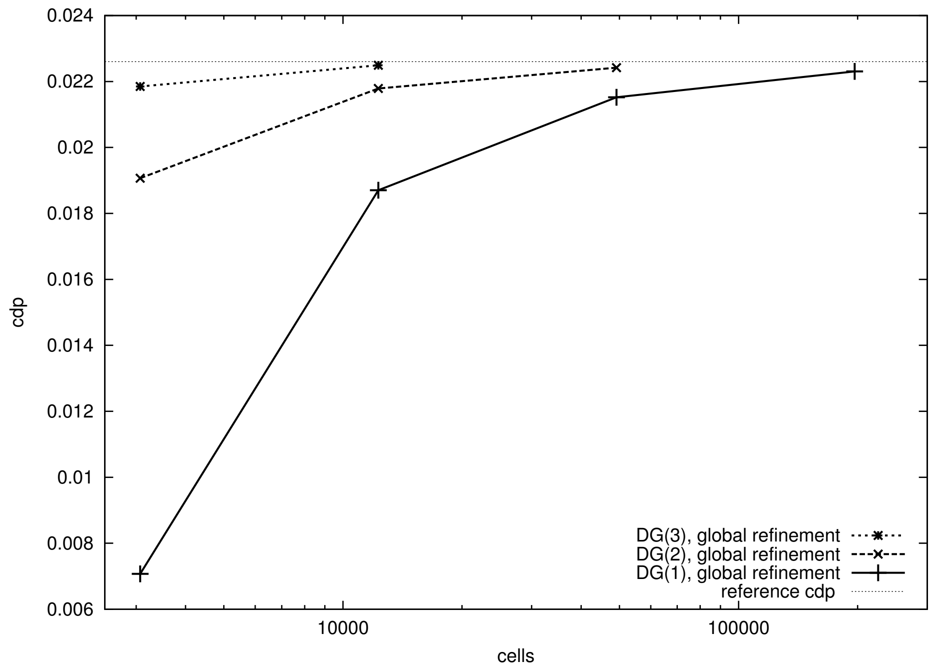


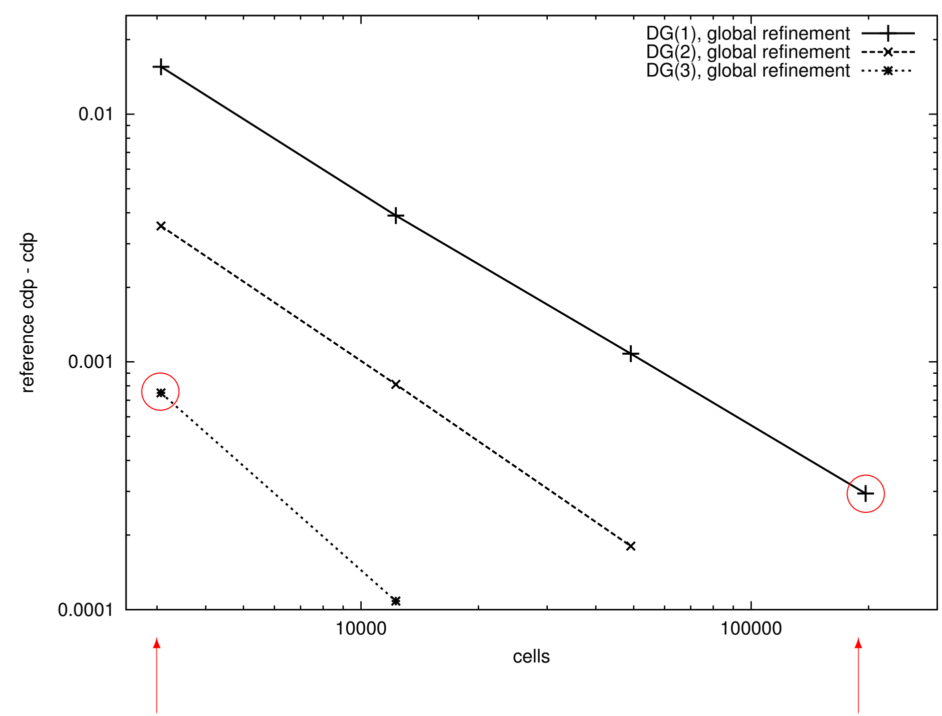
3.072 cells **196.608 cells**

Convergence of cdp

cdp



reference cdp - cdp

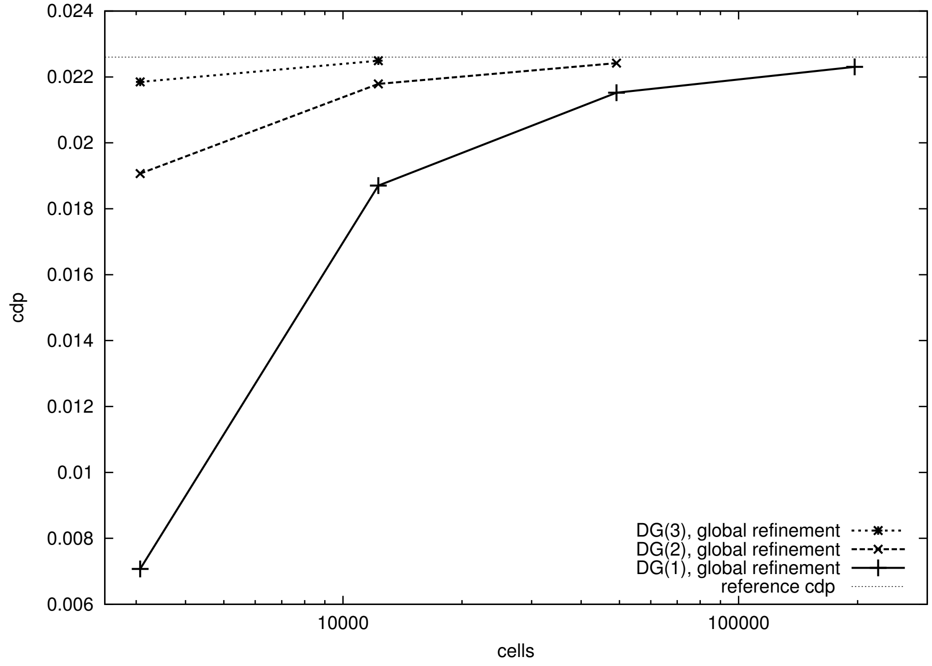


3.072 cells
196.608 dof

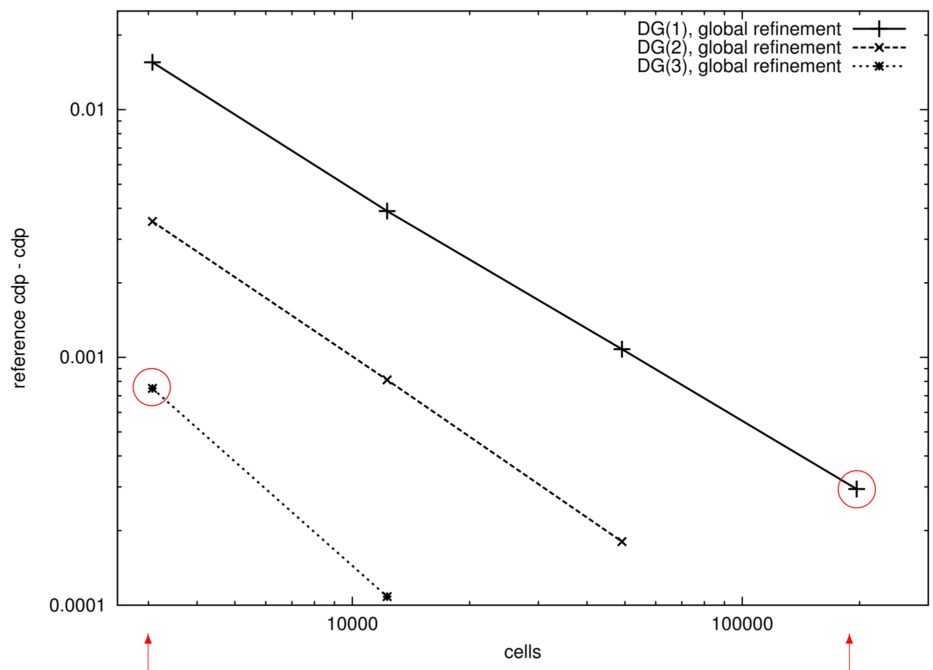
196.608 cells
3.145.728 dof

Convergence of cdp

cdp



reference cdp - cdp

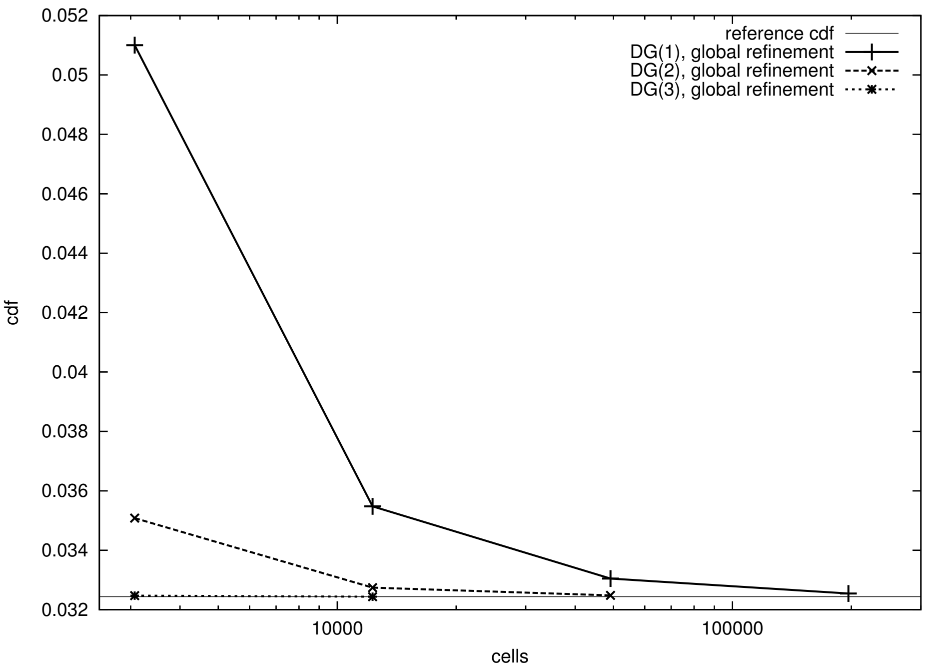


3.072 cells
196.608 dof
8 min

196.608 cells
3.145.728 dof
136 min

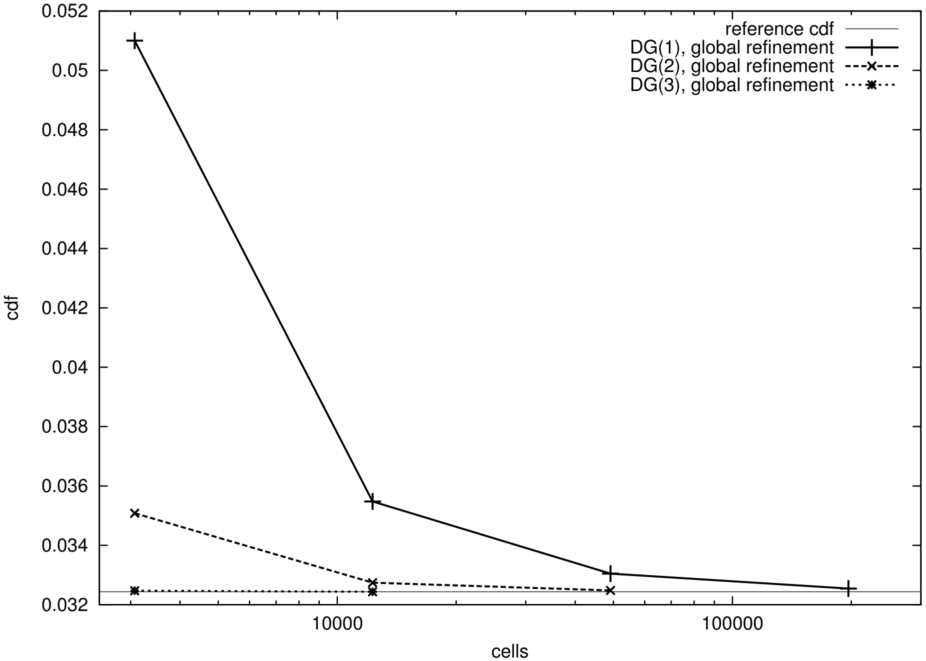
Convergence of cdf

cdf

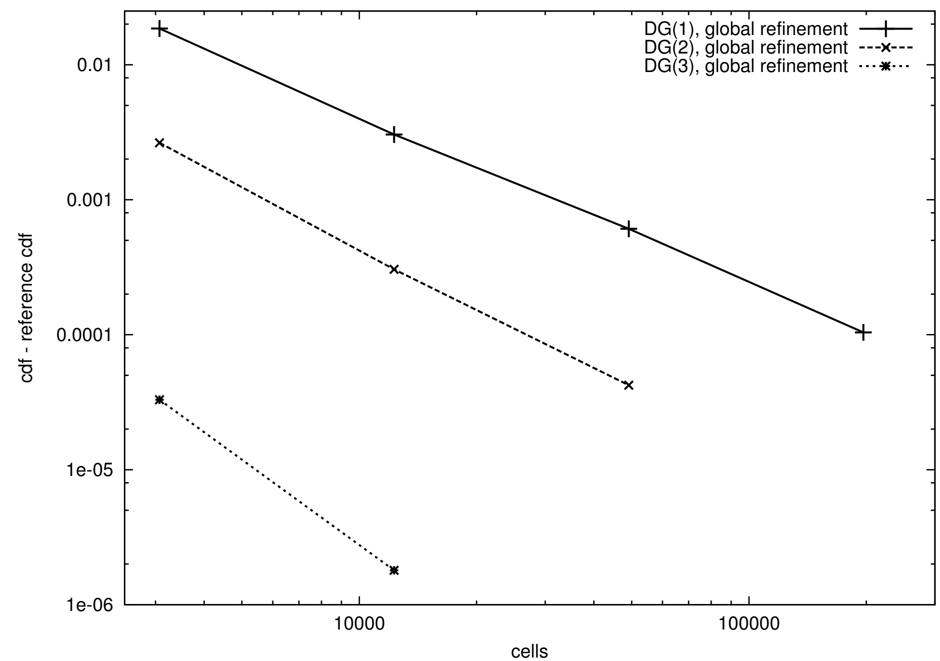


Convergence of cdf

cdf

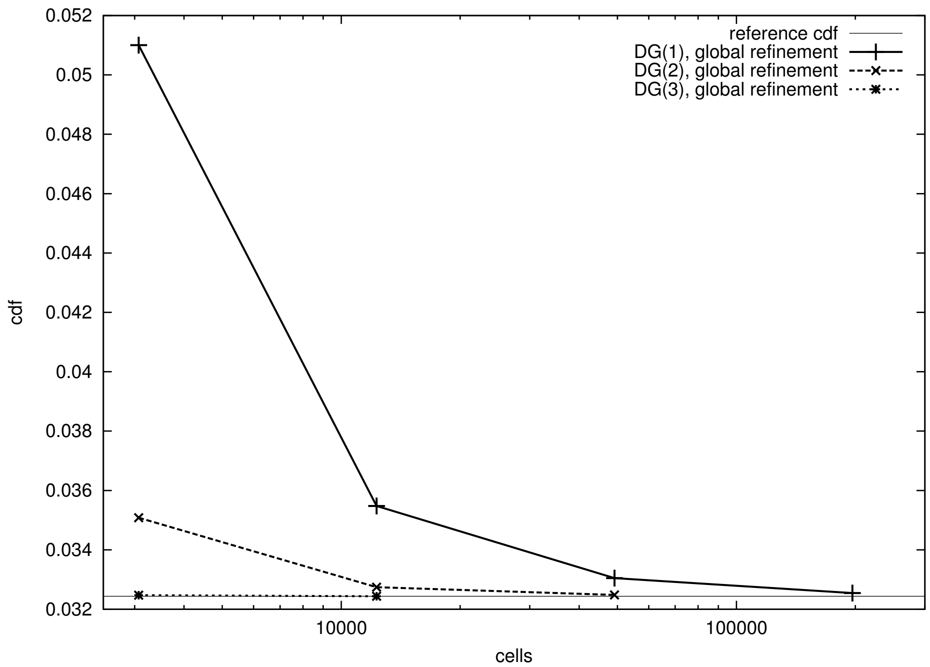


cdf - reference cdf

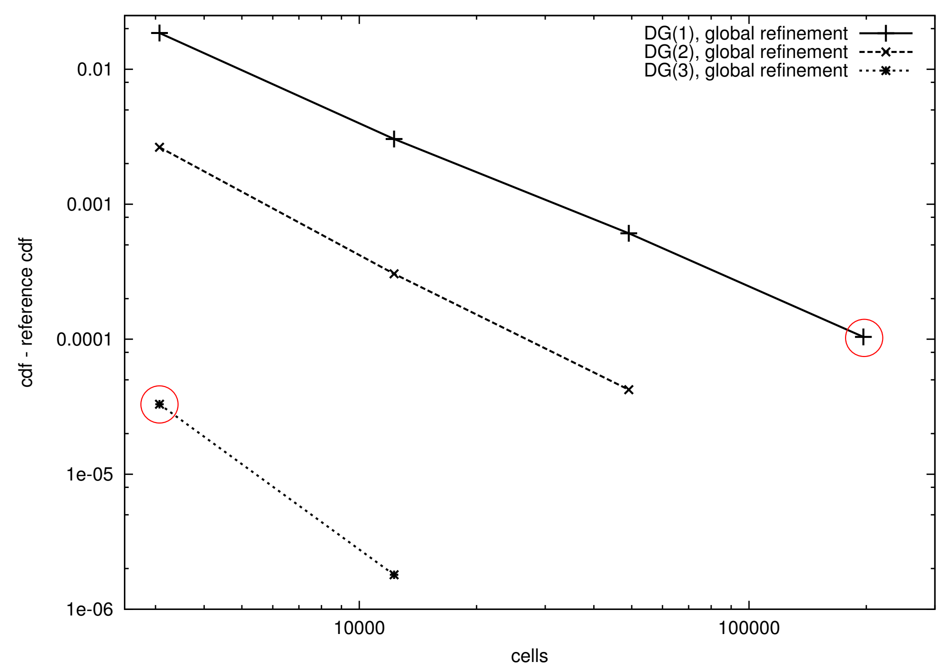


Convergence of cdf

cdf

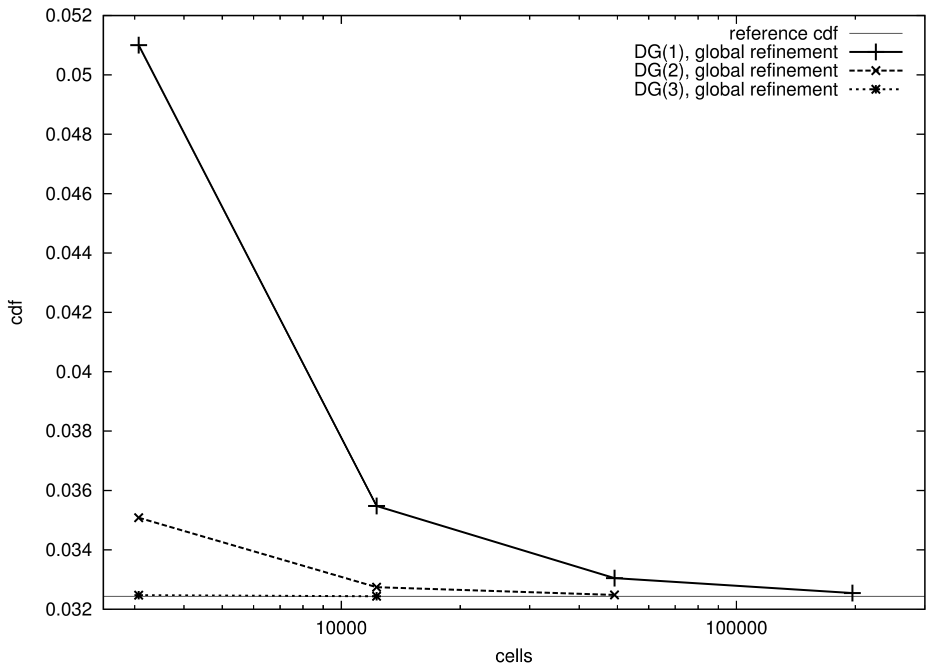


cdf - reference cdf

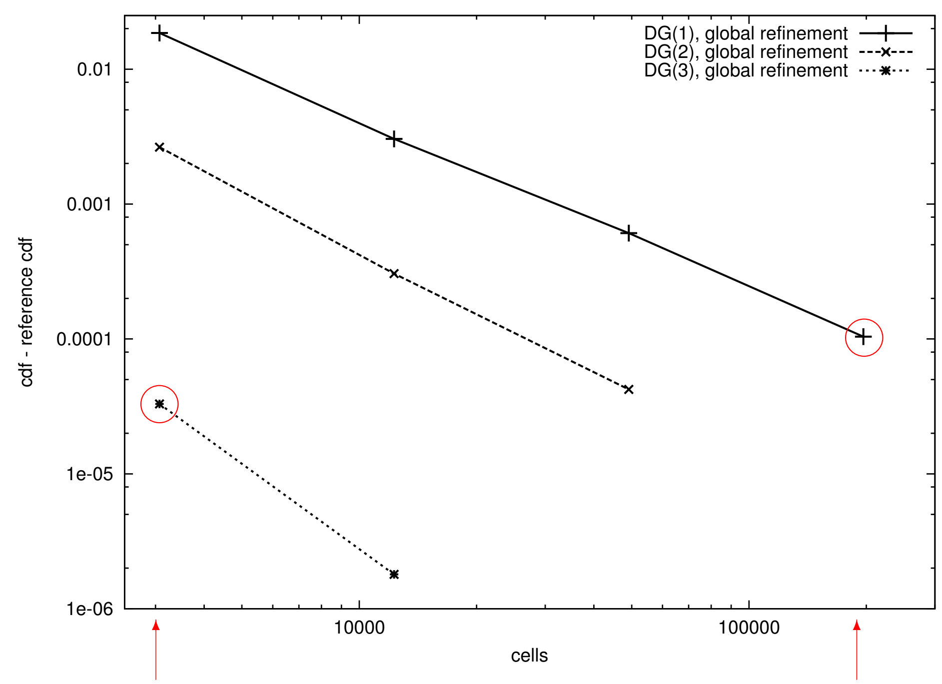


Convergence of cdf

cdf

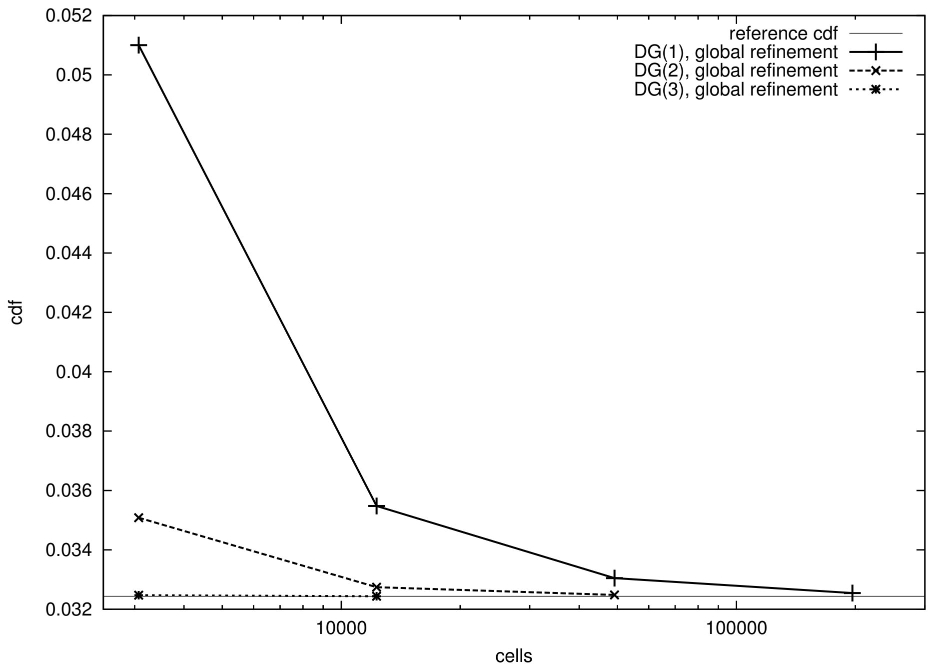


cdf - reference cdf

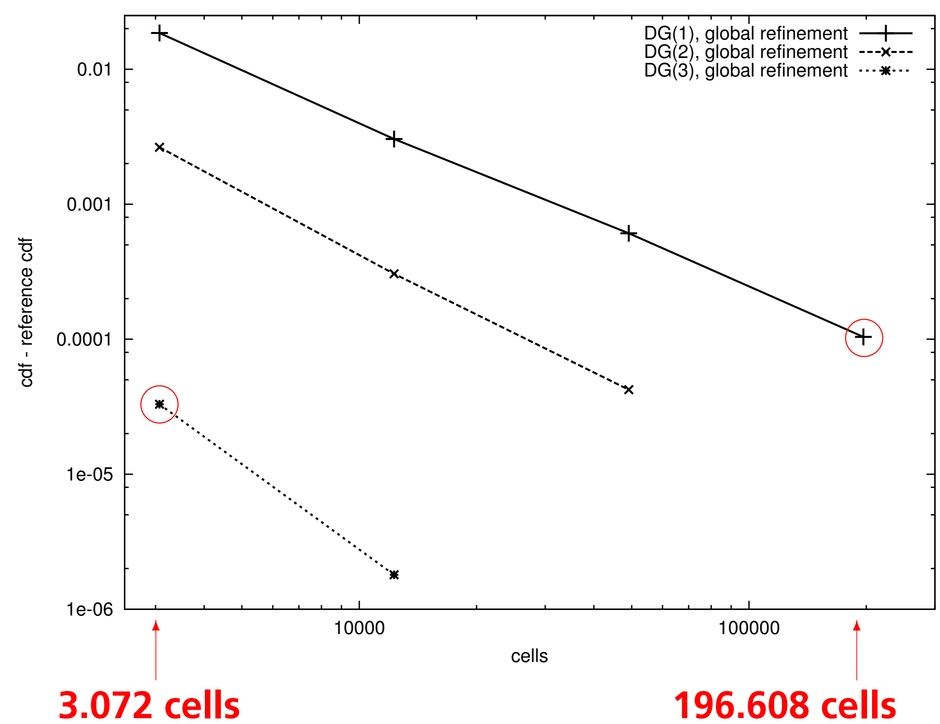


Convergence of cdf

cdf

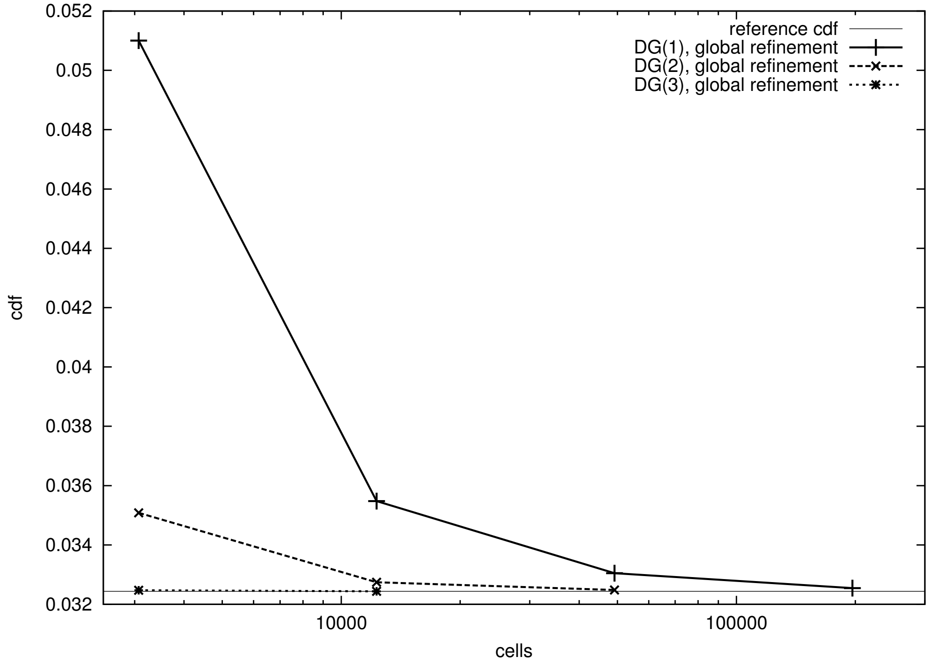


cdf - reference cdf

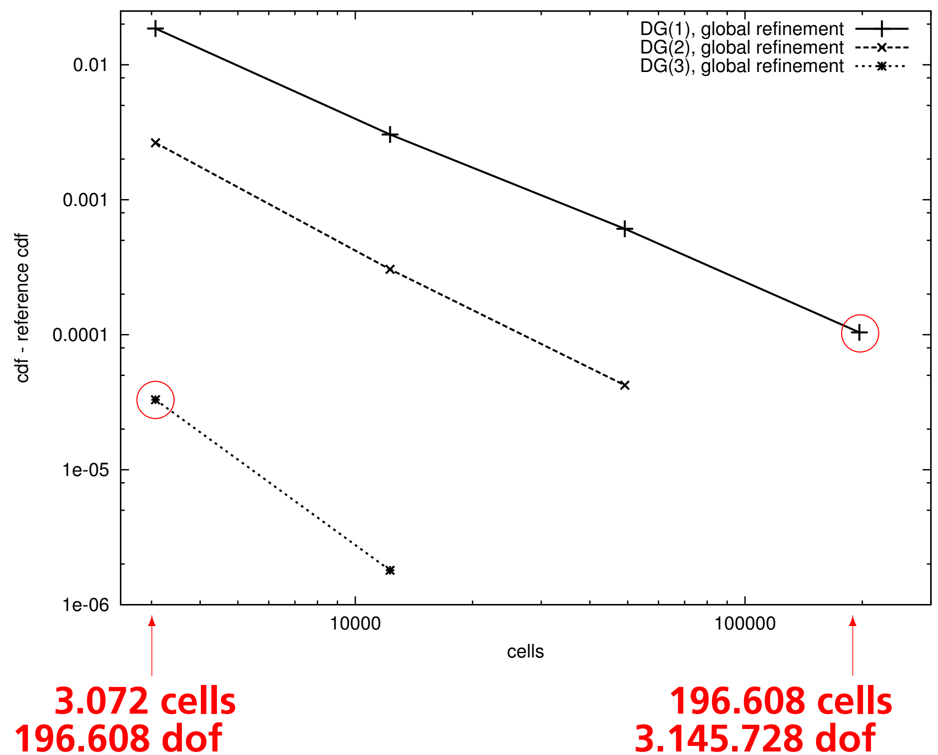


Convergence of cdf

cdf

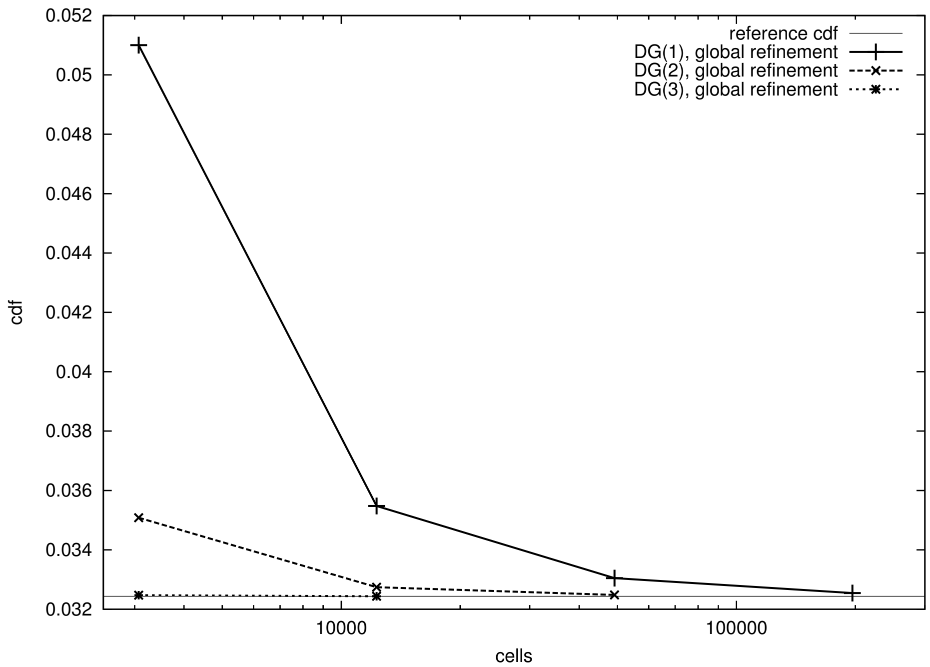


cdf - reference cdf

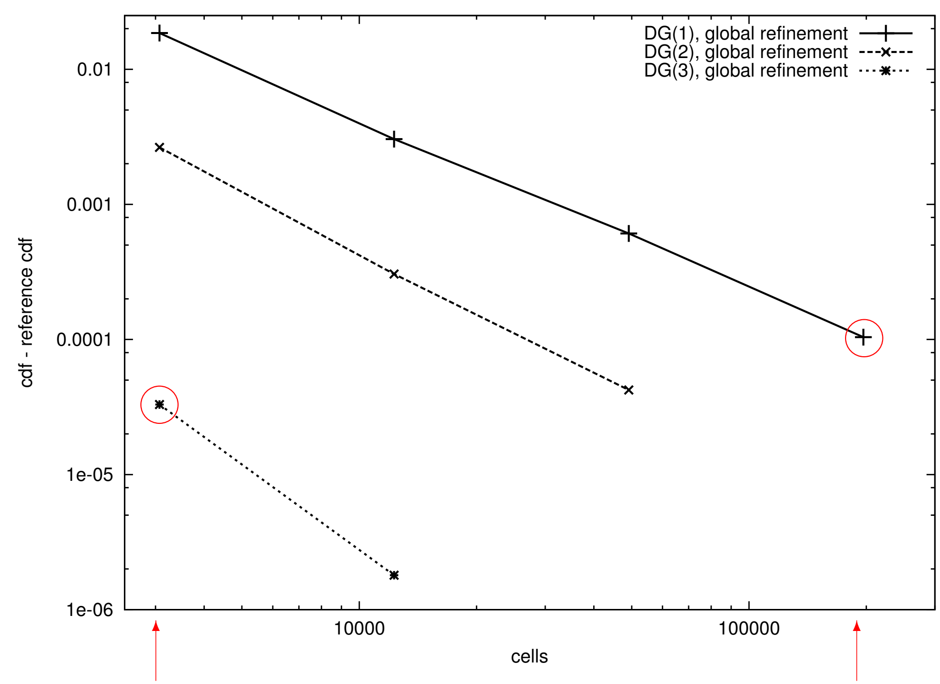


Convergence of cdf

cdf



cdf - reference cdf



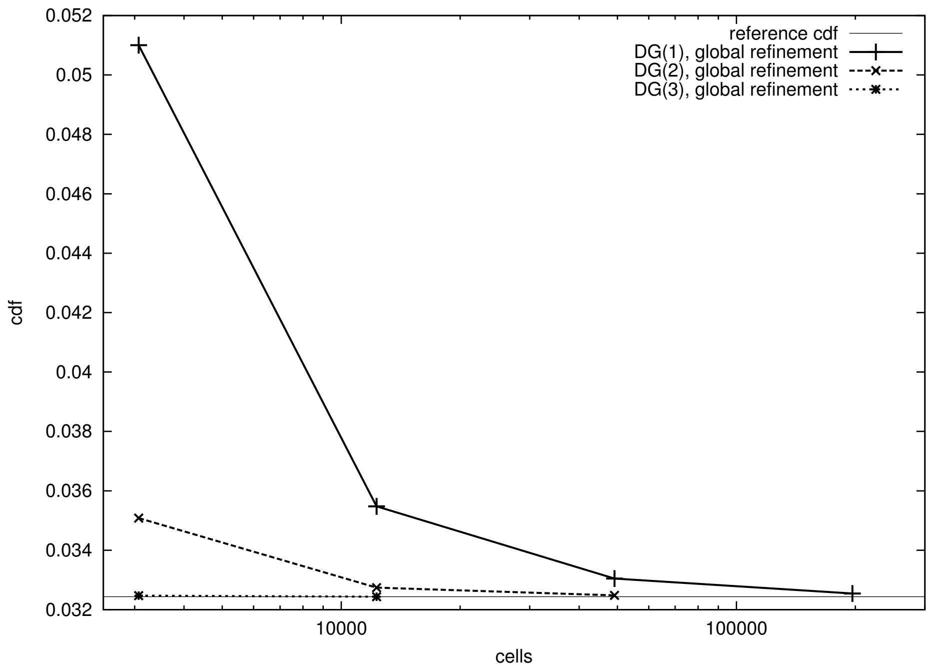
3.072 cells
196.608 dof
8 min

196.608 cells
3.145.728 dof
136 min

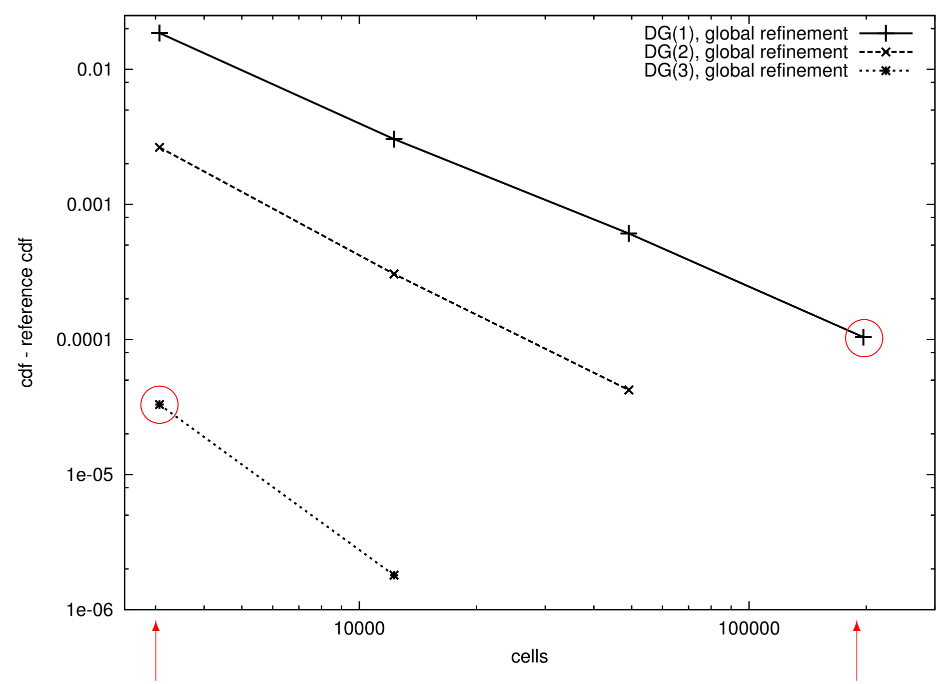
Deutsches Zentrum
für Luft- und Raumfahrt e.V.
in der Helmholtz-Gemeinschaft

Convergence of cdf

cdf



cdf - reference cdf



3.072 cells
196.608 dof
8 min

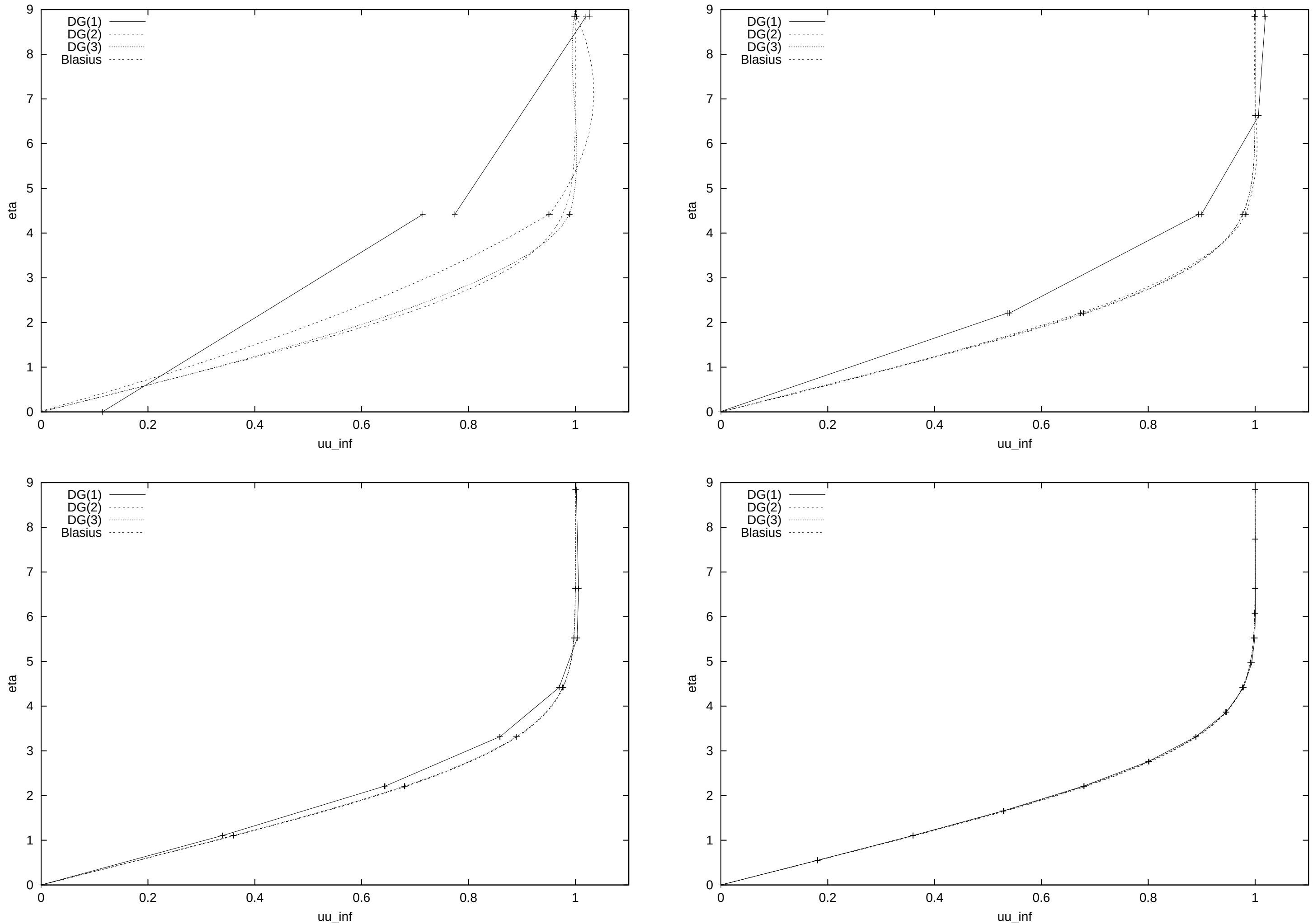
196.608 cells
3.145.728 dof
136 min

786.432 cells
12.582.912 dofs
≈ 12 h (extrapolated)

Higher order approximation of viscous boundary layers

Higher order approximation of viscous boundary layers

Flat plate problem: $M = 0.01$, $Re = 10000$



Higher order approximation of viscous boundary layers

Flat plate problem: $M = 0.01$, $Re = 10000$

Approximation of viscous force exerted on wall up to 5% requires

	DG(1)	DG(2)	DG(3)
elements	36	5	3
DoF	72	15	12

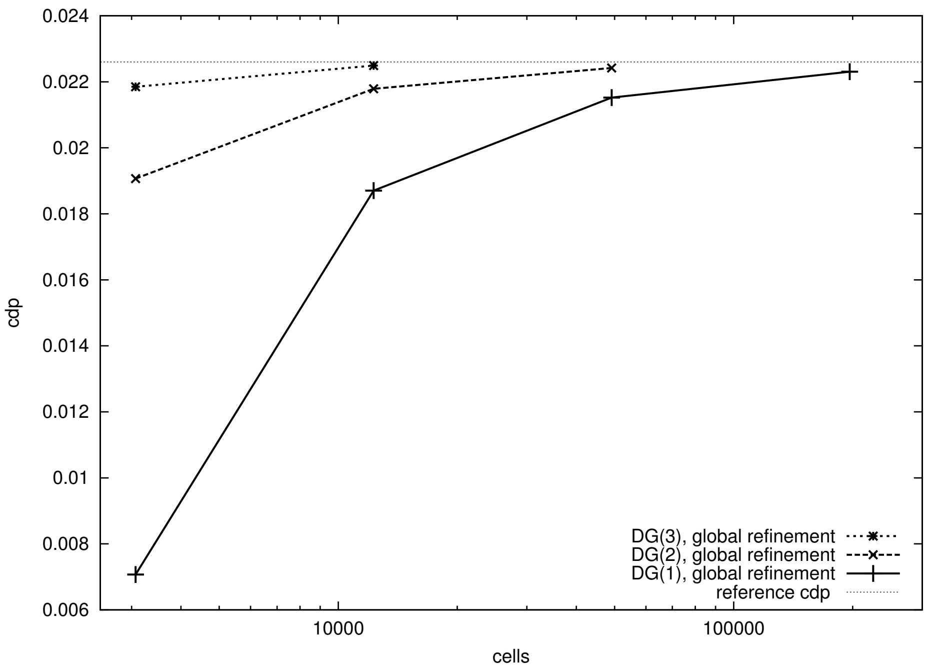
orthogonal to the wall

Local refinement: residual based

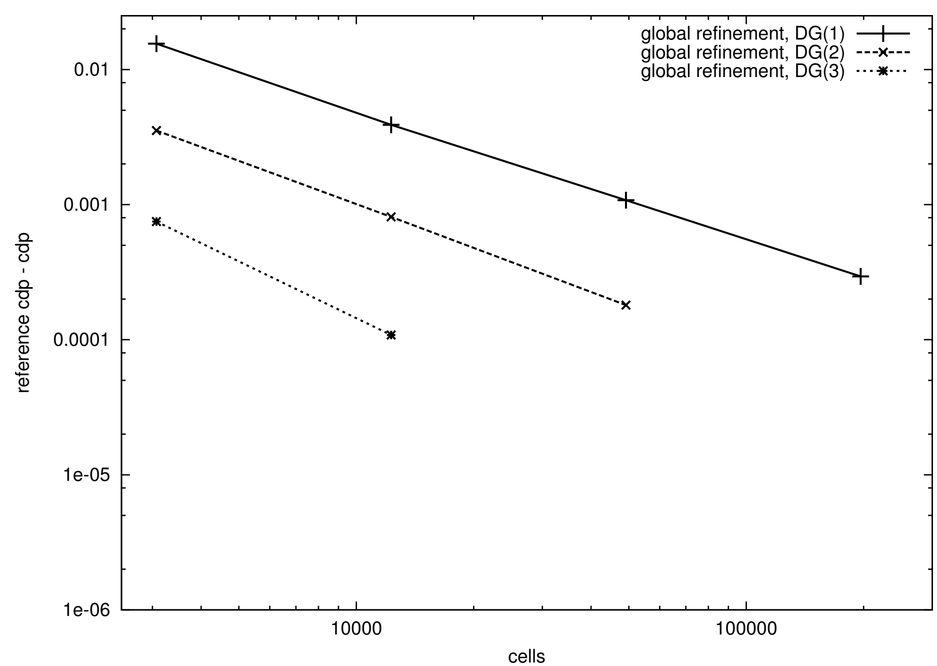
Local refinement: residual based

Test case: NACA0012, $M = 0.5$, $\alpha = 0$, $Re = 5000$

cdp



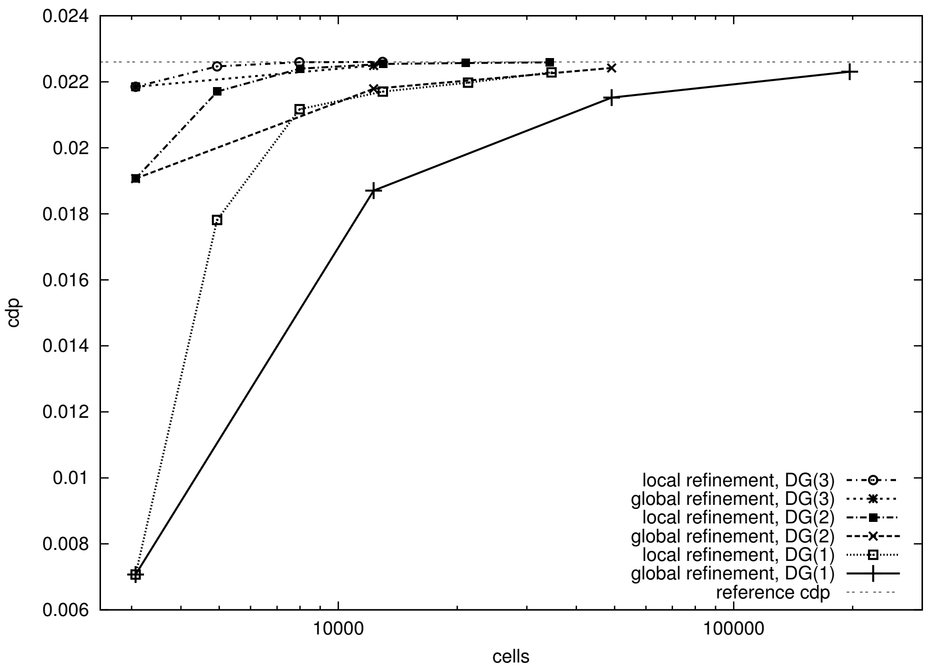
reference cdp - cdp



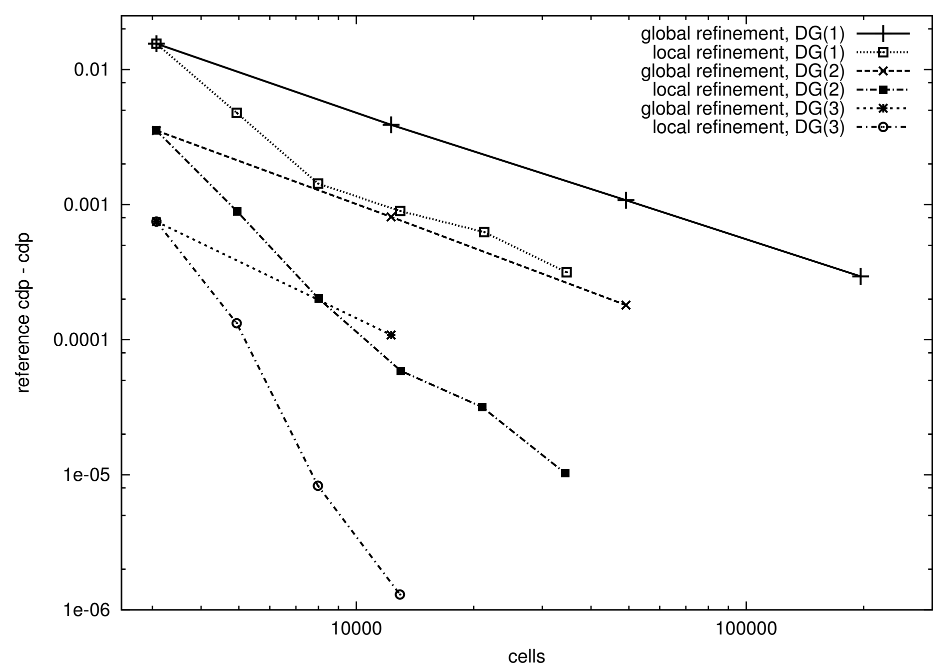
Local refinement: residual based

Test case: NACA0012, $M = 0.5$, $\alpha = 0$, $Re = 5000$

cdp



reference cdp - cdp



Duality-based *a posteriori* error estimation

Error representation with respect to a target quantity $J(\mathrm{u})$:

$$J(\mathbf{u}) - J(\mathbf{u}_h) = -\mathcal{N}(\mathbf{u}_h, \mathbf{z} - \mathbf{z}_h),$$

where $\mathrm{z}_h \in \mathrm{V}_{h,p}$, and z is the solution to the *dual* or *adjoint* problem: find $\mathrm{z} \in \mathrm{V}$ such that

$$\mathcal{M}(\mathbf{u}, \mathbf{u}_h; \mathbf{w}, \mathbf{z}) = J(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}.$$

where $\mathcal{M}(\mathrm{u}, \mathrm{u}_h; \cdot, \cdot)$ denotes the mean–value linearization

$$\begin{aligned} \mathcal{M}(\mathbf{u}, \mathbf{u}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{v}) &= \mathcal{N}(\mathbf{u}, \mathbf{v}) - \mathcal{N}(\mathbf{u}_h, \mathbf{v}) \\ &= \int_0^1 \mathcal{N}'[\theta\mathbf{u} + (1-\theta)\mathbf{u}_h](\mathbf{u} - \mathbf{u}_h, \mathbf{v})\, \mathrm{d}\theta \end{aligned}$$

Duality-based *a posteriori* error estimation

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***Linearized* dual problem: find $\tilde{\mathrm{z}} \in \mathrm{V}$ such that**

$$\mathcal{M}(\mathbf{u}_h, \mathbf{u}_h; \mathbf{w}, \tilde{\mathbf{z}}) = \mathcal{N}'[\mathbf{u}_h](\mathbf{w}, \tilde{\mathbf{z}}) = J(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}.$$

***Discretized* dual problem: find $\tilde{\mathrm{z}}_h \in \mathrm{V}_{h,\tilde{p}}$ such that**

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}, \tilde{\mathbf{z}}_h) = J(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{V}_{h,\tilde{p}}.$$

Error estimation and goal-oriented mesh refinement

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx -\mathcal{N}(\mathbf{u}_h, \tilde{\mathbf{z}}_h - \mathbf{z}_h)$$

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx \sum_{\kappa \in \mathcal{T}_h} \left\{ \int_\kappa R(\mathbf{u}_h)\,(\tilde{\mathbf{z}}_h - \mathbf{z}_h)\ \mathrm{d}\mathbf{x} + \int_{\partial\kappa} r(\mathbf{u}_h)\,(\tilde{\mathbf{z}}_h - \mathbf{z}_h)\ \mathrm{d}s \right\} =: \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa =: \eta$$

Error estimation and goal-oriented mesh refinement

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx -\mathcal{N}(\mathbf{u}_h, \tilde{\mathbf{z}}_h - \mathbf{z}_h)$$

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$J(\mathbf{u})$	**target quantity, e.g.** $J(\mathbf{u}) = c_d$, $J(\mathbf{u}) = c_l$, **etc.**
$R(\mathbf{u}_h)$	**cell residuals**
$r(\mathbf{u}_h)$	**face residuals**
$\tilde{\mathbf{z}}_h \in \mathbf{V}_{h,\tilde{p}}$	**computed dual/adjoint solution**
$\mathbf{z}_h \in \mathbf{V}_{h,p}$	**any discrete function in** $\mathrm{V}_{h,p}$**, e.g. interpolate of** $\tilde{\mathrm{z}}_h$**, or simply** $\mathrm{z}_h = 0$
η_κ	**dual-weighted residual indicator used for adjoint-based refinement**
$\eta = \sum_{\kappa\in\mathcal{T}_h} \eta_\kappa$	**estimate of the true error** $J(\mathrm{u}) - J(\mathrm{u}_h)$

Error estimation and goal-oriented mesh refinement

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx -\mathcal{N}(\mathbf{u}_h, \tilde{\mathbf{z}}_h - \mathbf{z}_h)$$

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$R(\mathbf{u}_h)$	**cell residuals**
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η_κ	**dual-weighted residual indicator used for adjoint-based refinement**
$\eta = \sum_{\kappa\in\mathcal{T}_h} \eta_\kappa$	**estimate of the true error** $J(\mathbf{u}) - J(\mathbf{u}_h)$

The error estimate η **can be employed to improve the computed value** $J(\mathbf{u}_h)$**:**

$$\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_h) + \eta$$

Error estimation and goal-oriented refinement

Viscous flow at $M = 0.5$, $Re = 5000$, $\alpha = 0$ around NACA0012 airfoil

Target quantity: $J(\mathbf{u}) = c_{dp}$

# cells	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\eta = \sum_K \eta_K$	θ
3072	1.55e-02	1.06e-02	0.68
4941	4.48e-03	3.91e-03	0.87
8133	1.28e-03	1.17e-03	0.92
13521	3.33e-04	3.15e-04	0.95
21972	8.94e-05	8.75e-05	0.98
35823	3.15e-05	3.19e-05	1.01

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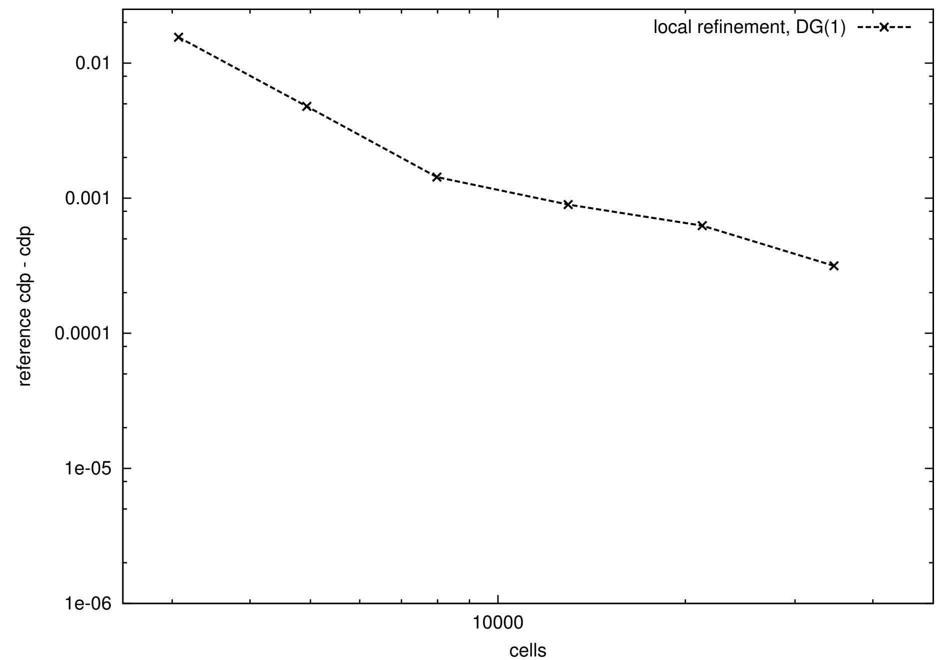
reference cdp - cdp



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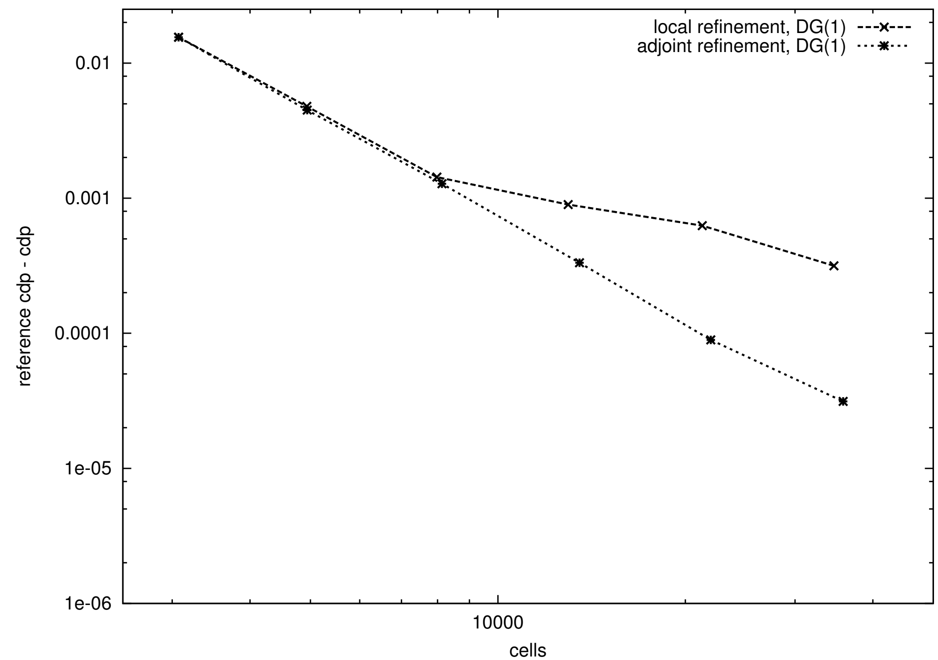

Error estimation and goal-oriented refinement

Viscous flow at $M = 0.5$, $Re = 5000$, $\alpha = 0$ around NACA0012 airfoil

Target quantity: $J(\mathbf{u}) = c_{dp}$

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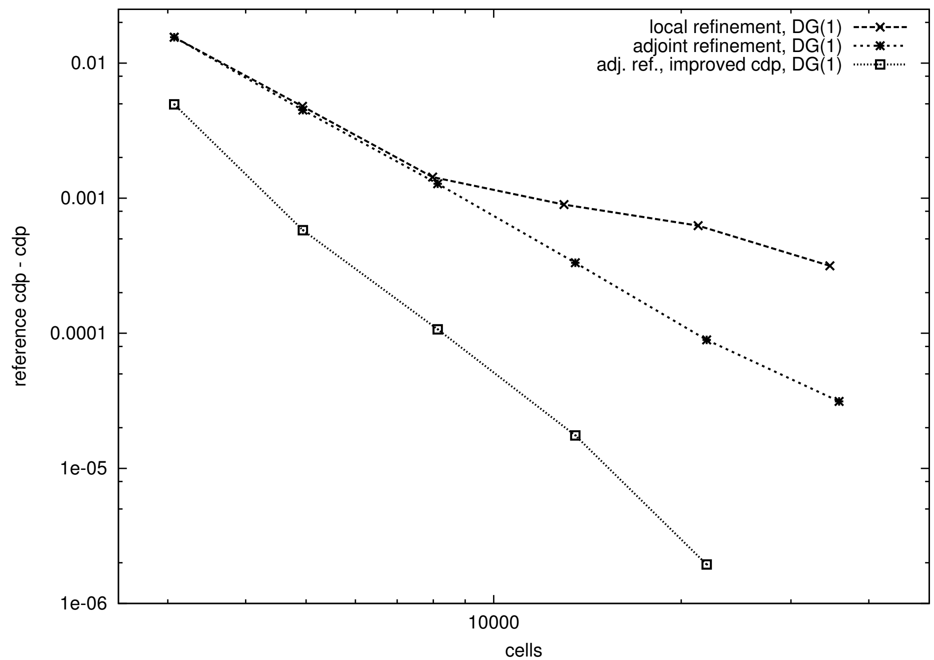


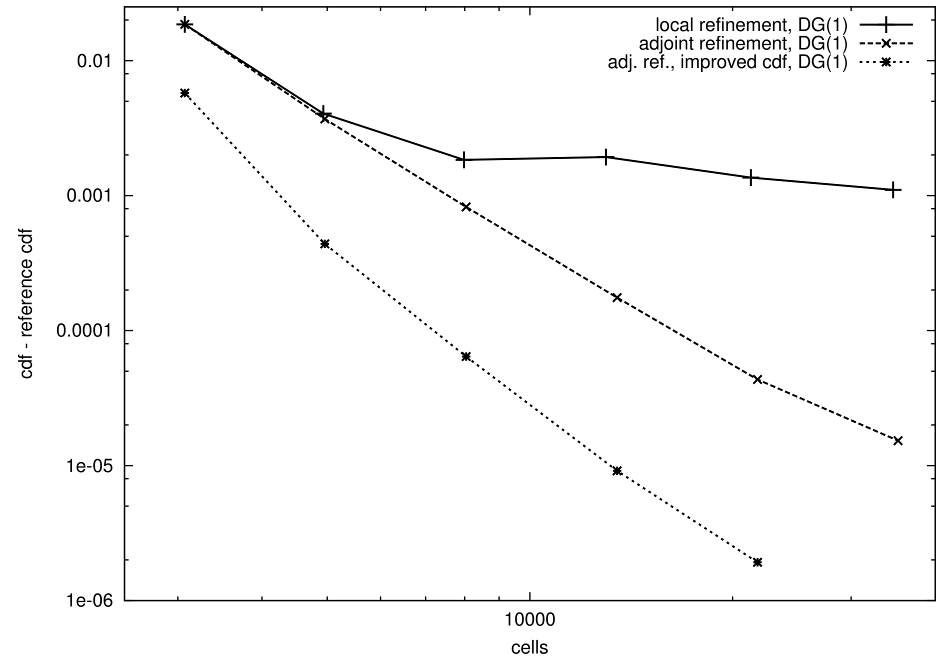
Error estimation and goal-oriented refinement

Viscous flow at $M = 0.5$, $Re = 5000$, $\alpha = 0$ around NACA0012 airfoil

Target quantity: $J(\mathbf{u}) = c_{df}$

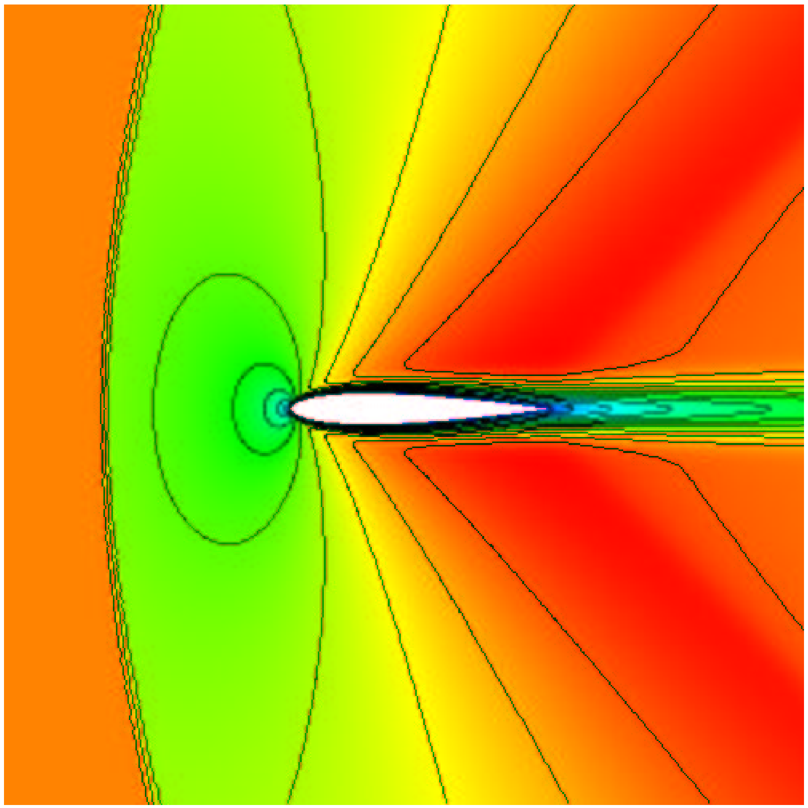
# cells	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\eta = \sum_K \eta_K$	θ
3072	-1.856e-02	-1.282e-02	0.69
4959	-3.699e-03	-3.260e-03	0.88
8040	-8.245e-04	-7.602e-04	0.92
13473	-1.756e-04	-1.665e-04	0.95
21783	-4.337e-05	-4.148e-05	0.96
35214	-1.534e-05	-1.542e-05	1.01

cdf - reference cdf

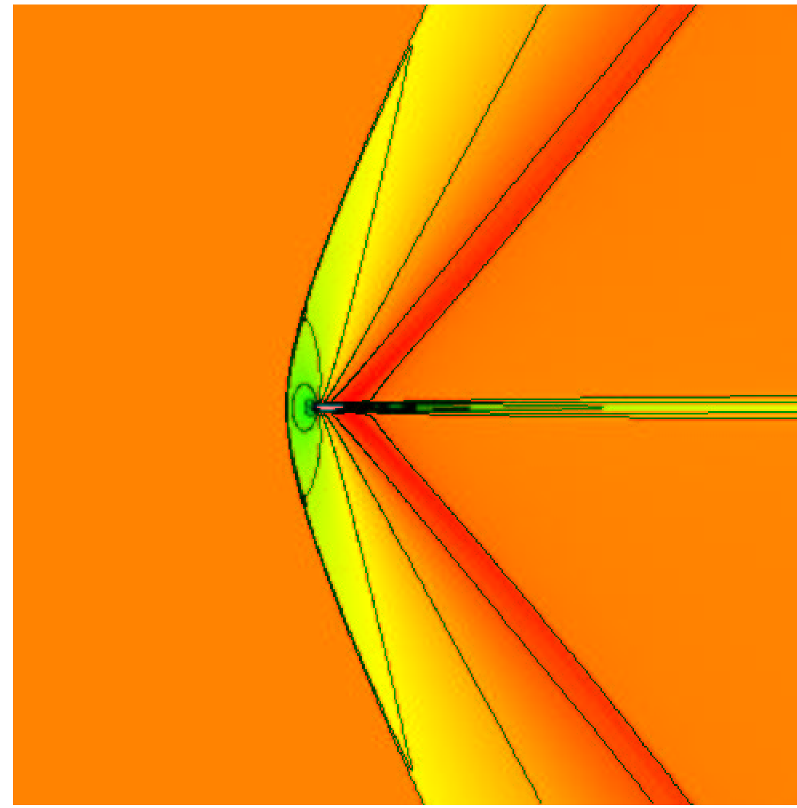


A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



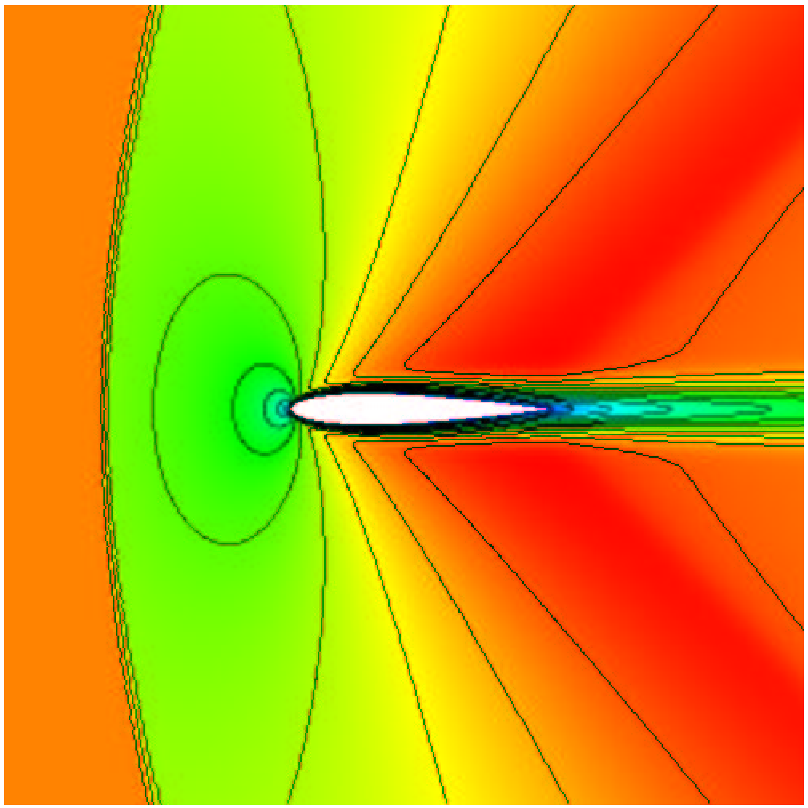
Mach isolines: close view



Mach isolines: distant view

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil

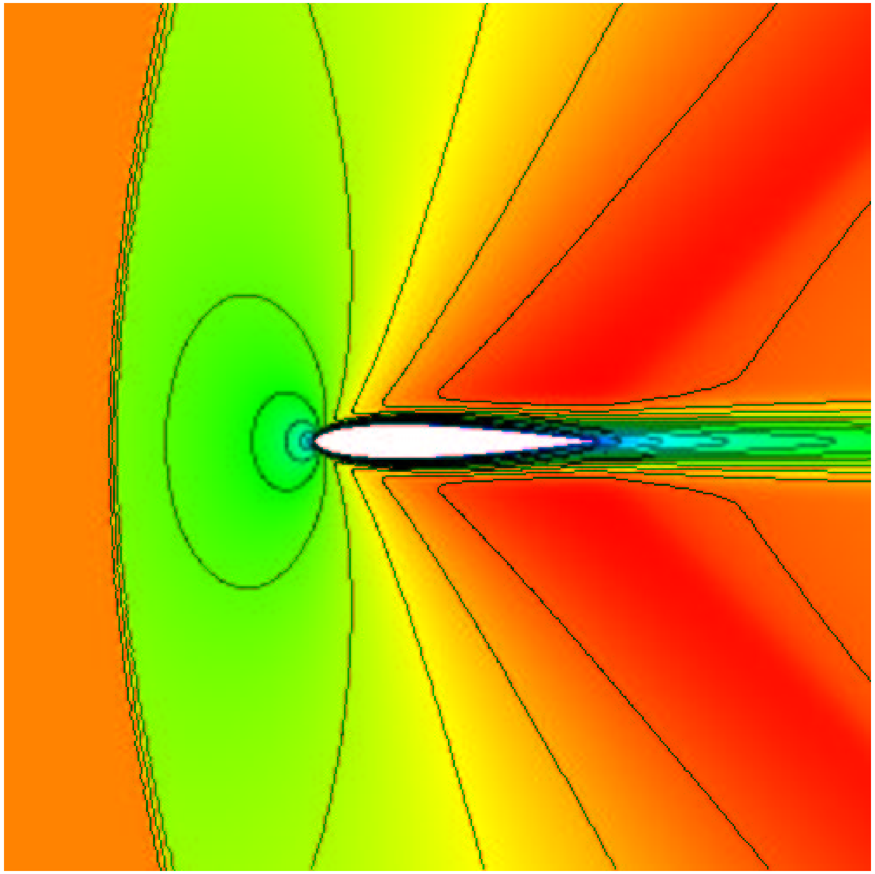


Target quantity: $J(\mathbf{u}) = c_{dp}$

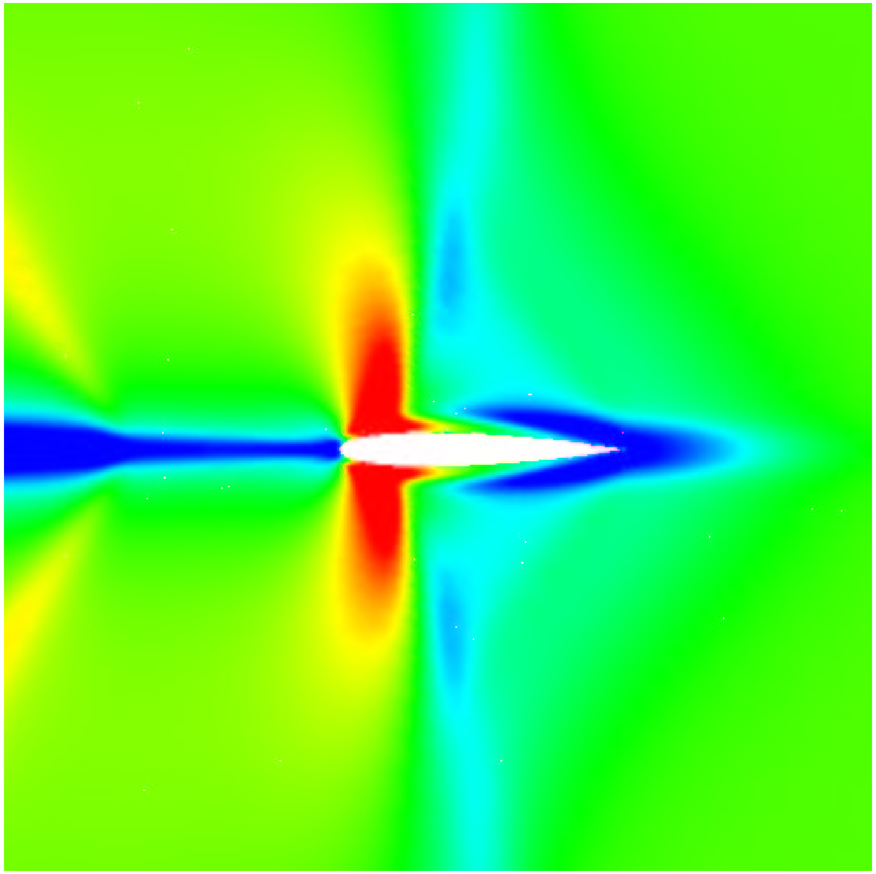
# cells	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\eta = \sum_K \eta_K$	θ
768	-1.363e-02	-6.312e-03	0.46
1260	-3.203e-03	-2.995e-03	0.94
2154	-4.844e-04	-5.368e-04	1.11
3570	-3.474e-04	-3.333e-04	0.96
6021	-1.835e-04	-1.856e-04	1.01
10038	-1.644e-04	-1.653e-04	1.01

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



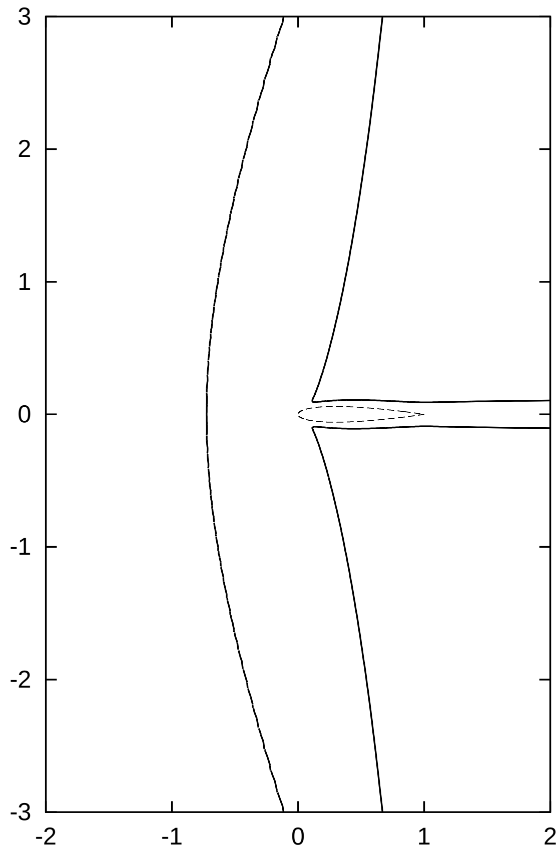
Mach isolines of primal solution

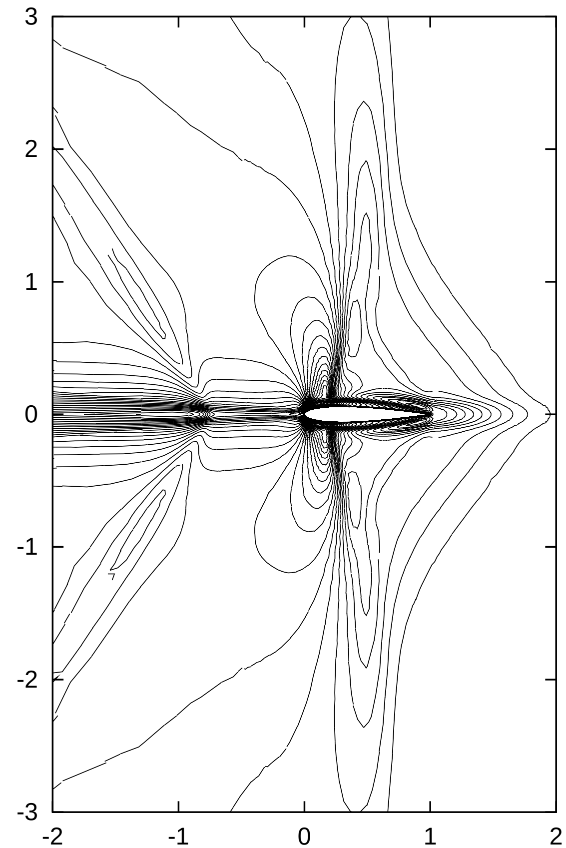


z_1 component of dual solution ($J(\mathrm{u}) = c_{dp}$)

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



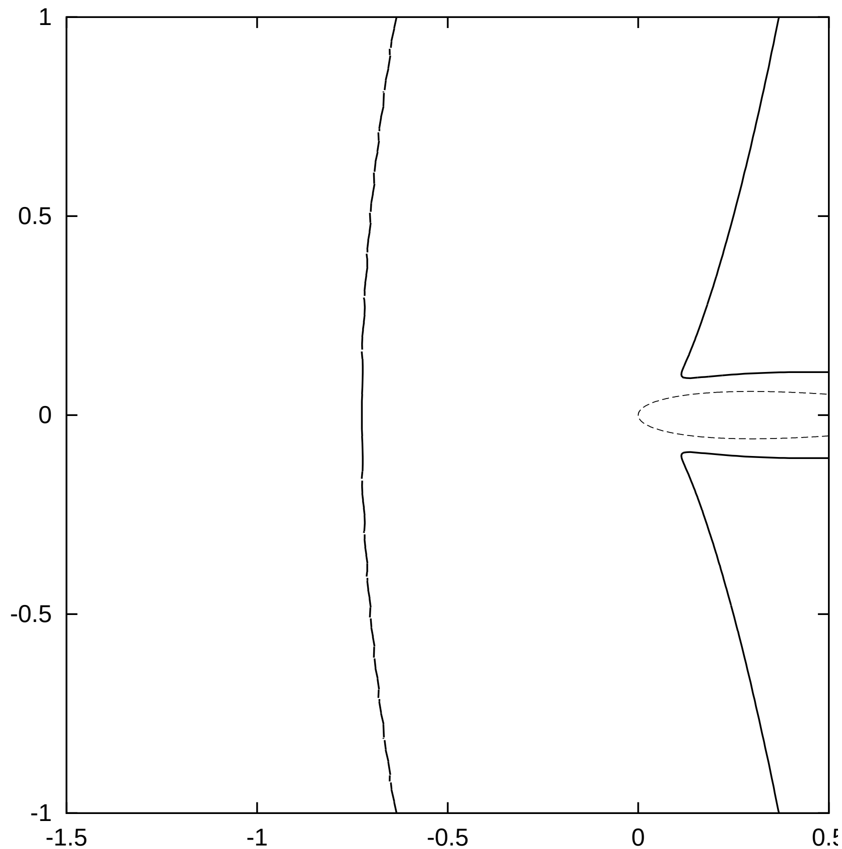


Sonic Mach isolines ($M = 1$) of primal solution

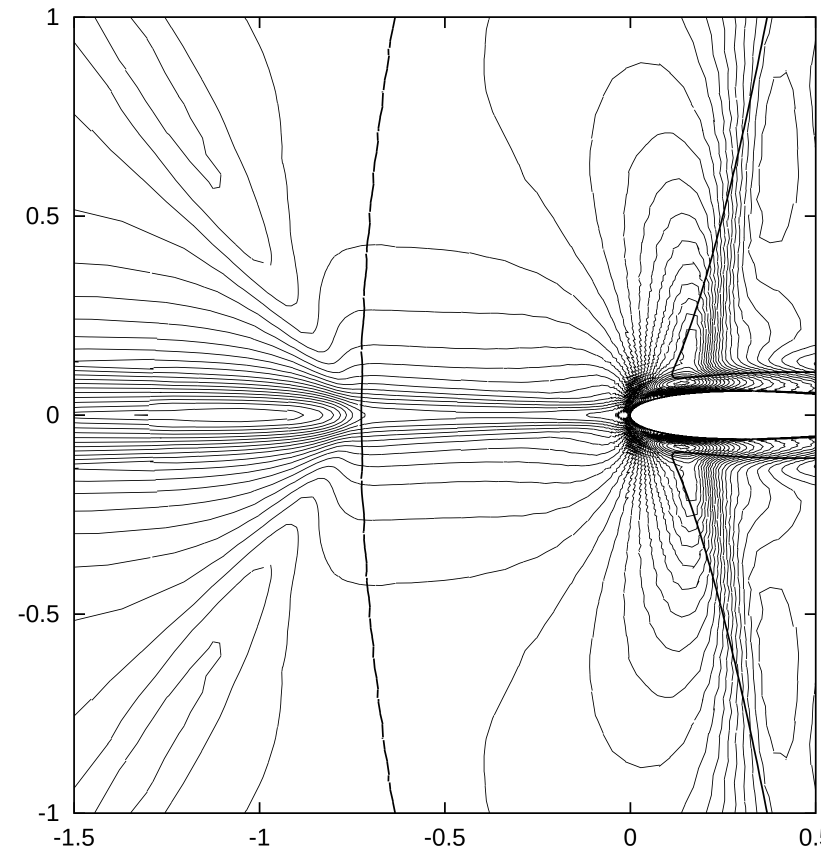
z_1 isolines of dual solution ($J(\mathrm{u}) = c_{dp}$)

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



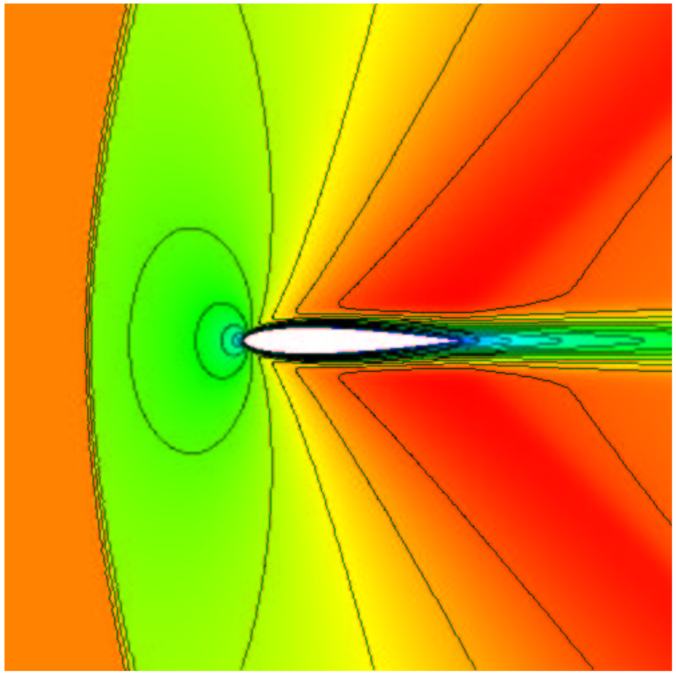
Sonic Mach isolines ($M = 1$) of primal solution



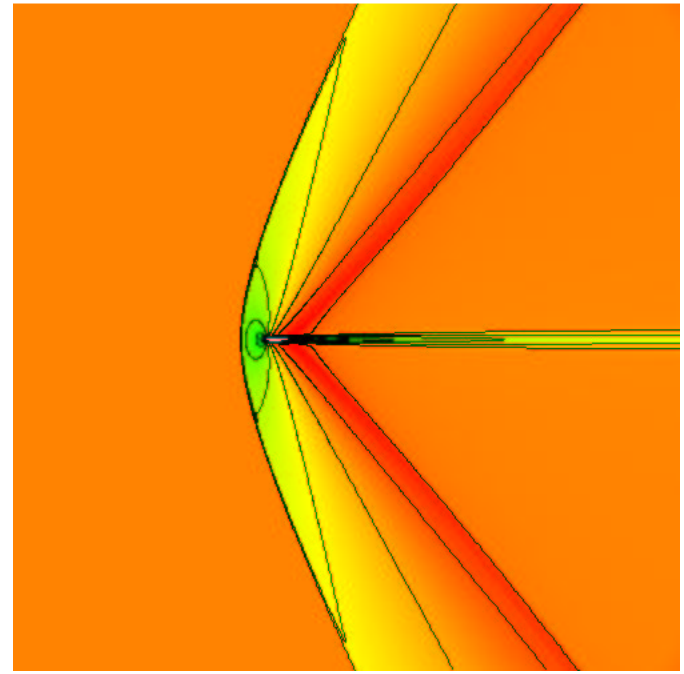
z_1 isolines of dual solution ($J(\mathrm{u}) = c_{dp}$)

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



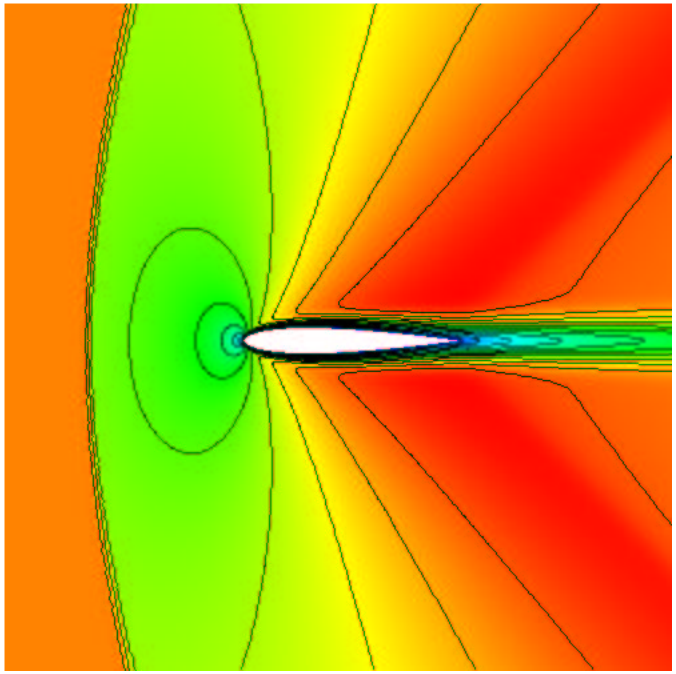
Mach isolines: close view



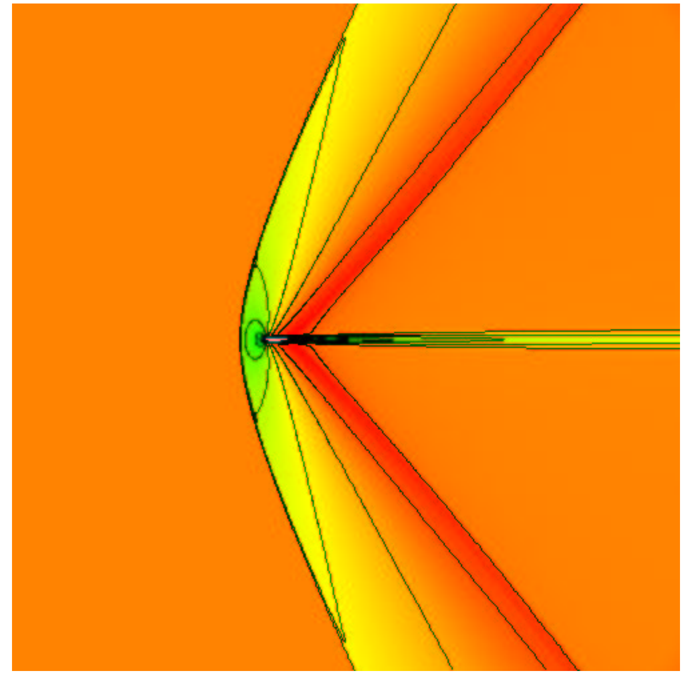
Mach isolines: distant view

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



Mach isolines: close view

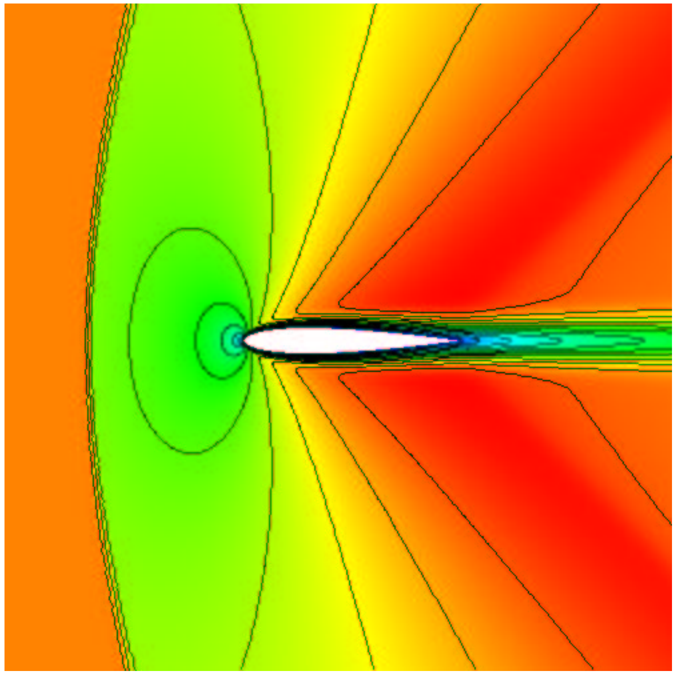


Mach isolines: distant view

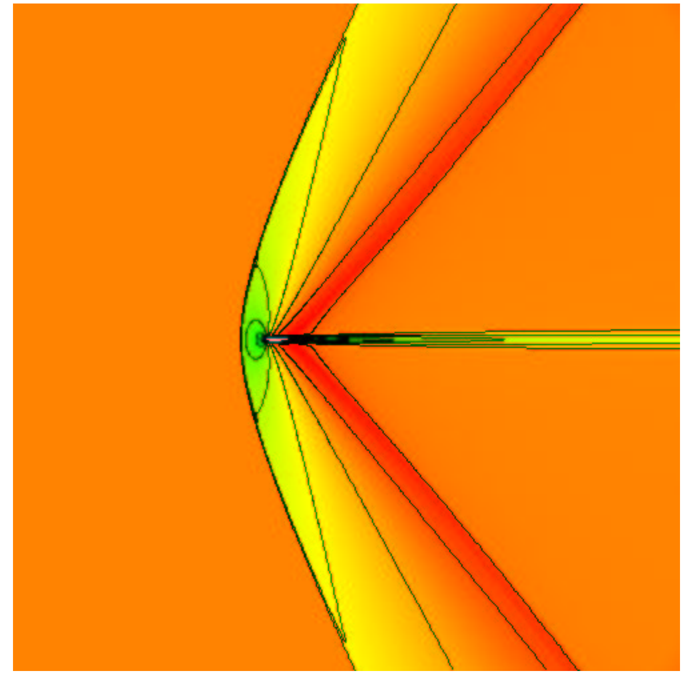
For the efficient and accurate approximation of $J(\mathrm{u}) = c_{dp}$:

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



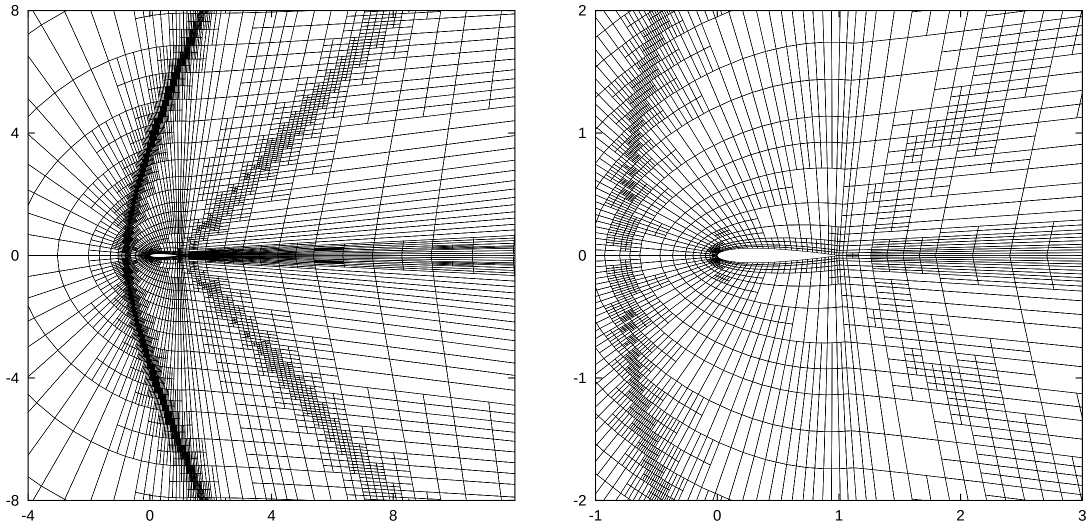
Mach isolines: close view



Mach isolines: distant view

For the efficient and accurate approximation of $J(\mathrm{u}) = c_{dp}$:

How should the mesh look like?


8
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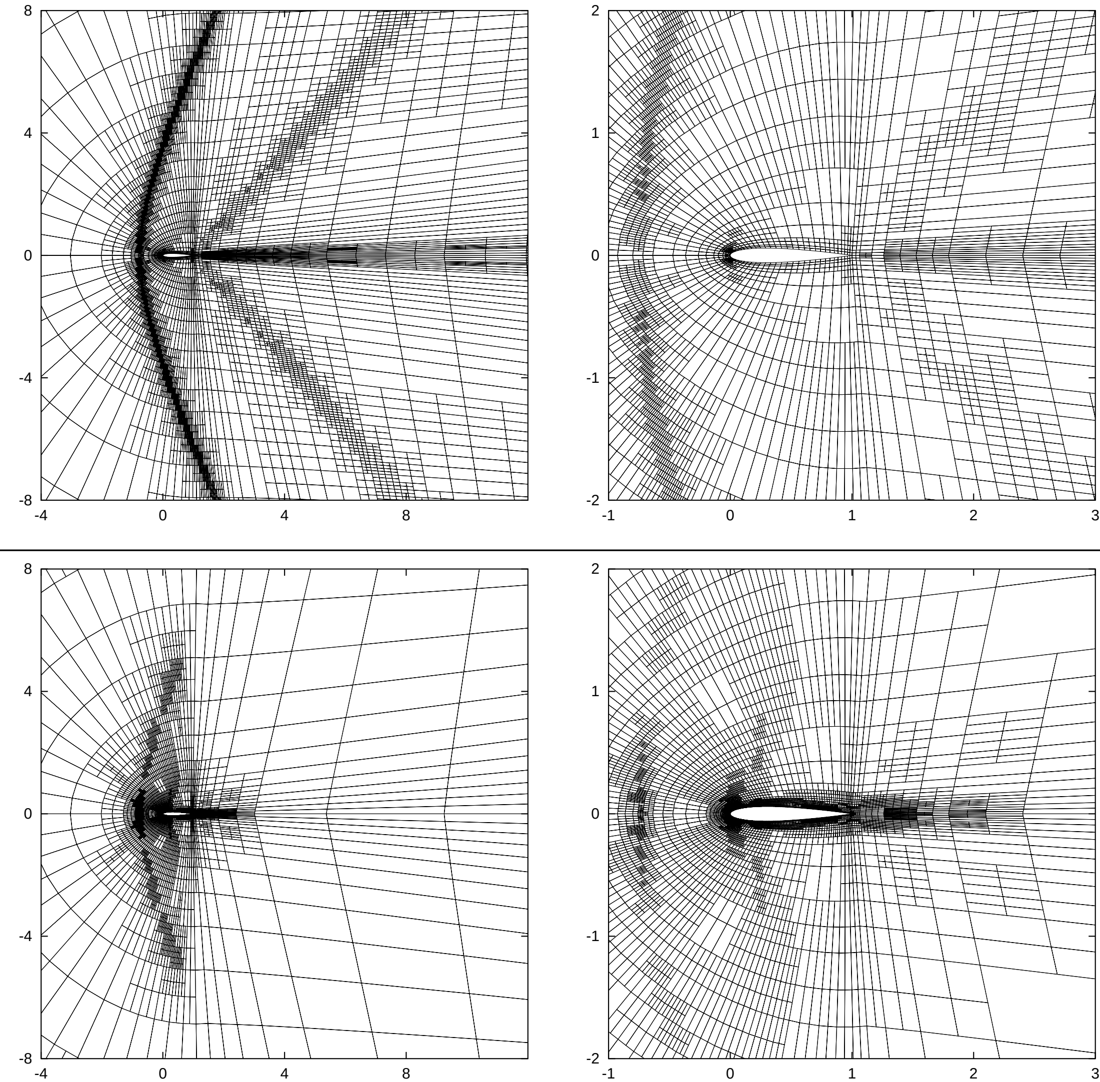
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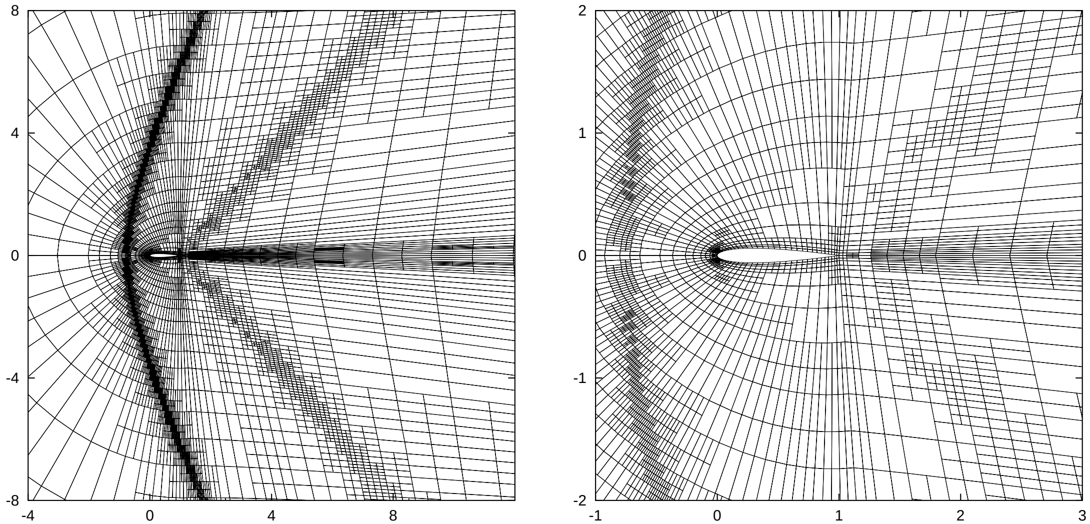
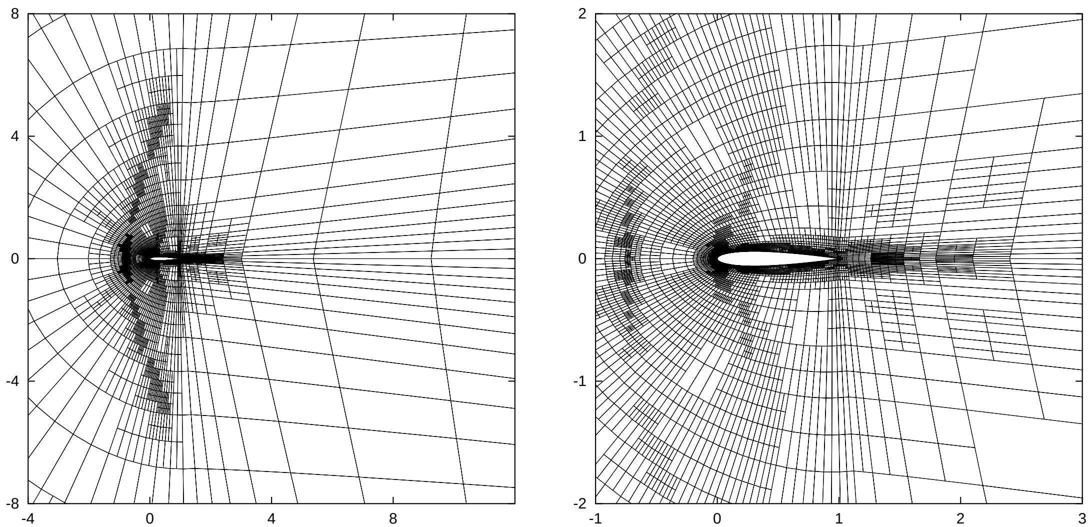
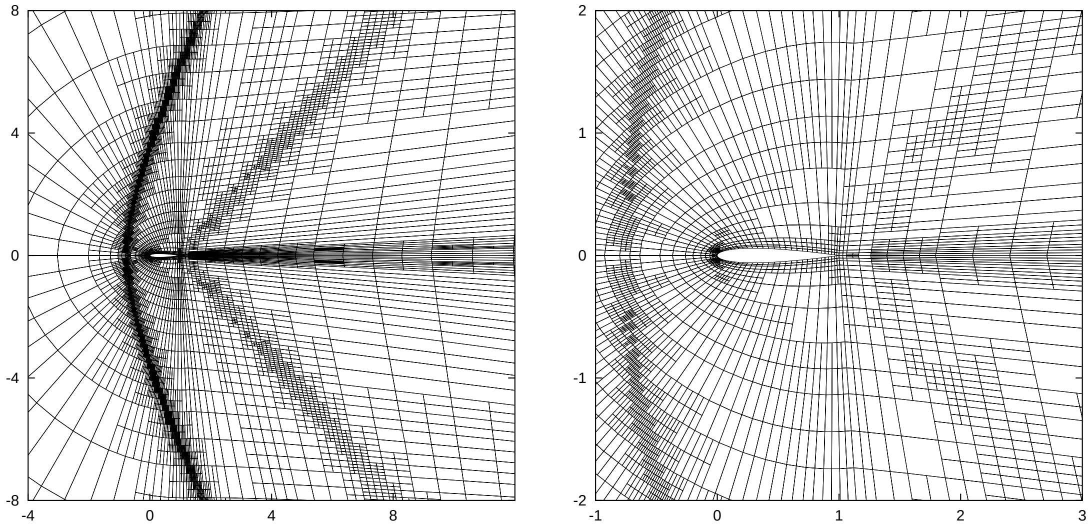




Residual-based refinement:
17670 cells with 282720 dofs
error in c_{dp} : $1.9 \cdot 10^{-3}$
error in c_{df} : $1.1 \cdot 10^{-2}$



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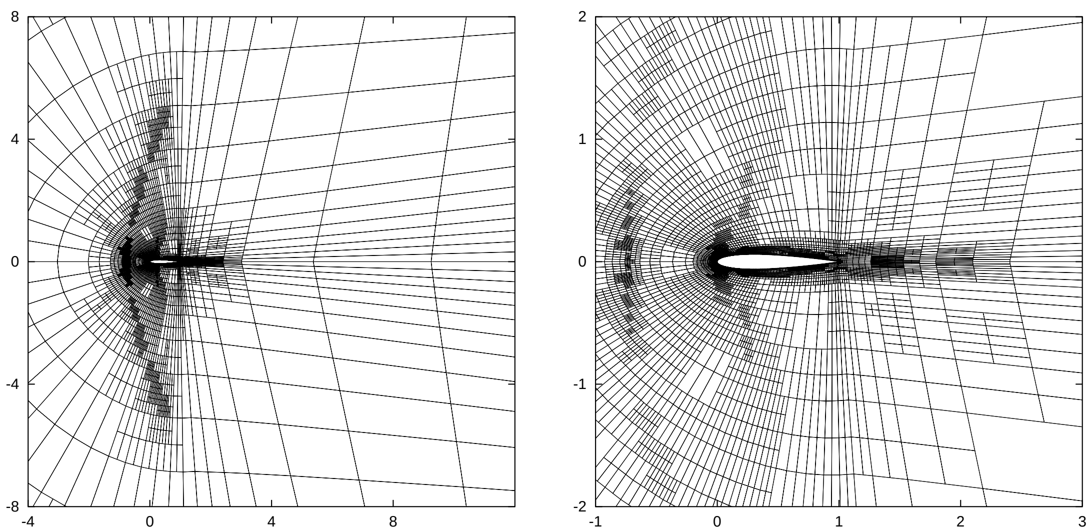


Goal-oriented refinement:
10038 cells with 160608 dofs
error in c_{dp} : $1.6 \cdot 10^{-4}$
error in c_{df} : $7.2 \cdot 10^{-4}$

Anisotropic refinement

Use a residual-based or adjoint-based indicator to select the elements to be refined

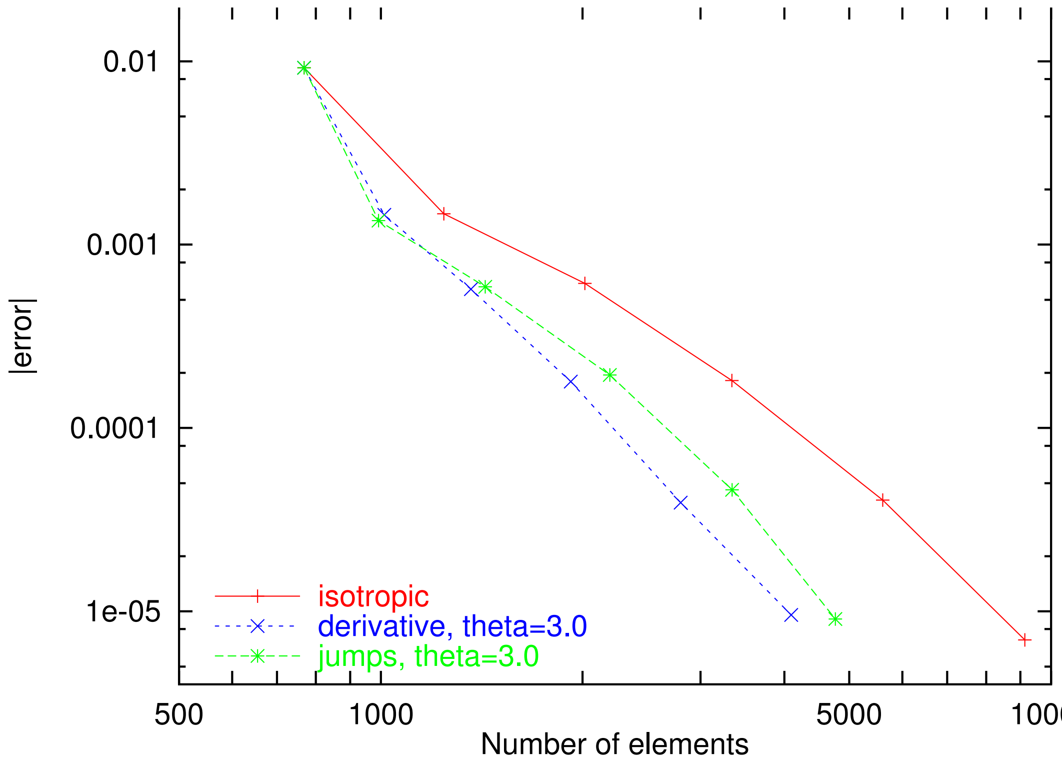
Use an anisotropic indicator to determine the anisotropic refinement case

Anisotropic indicators:

- Jump indicator: jump across a face is connected to approximation quality orthogonal to the face
- Derivative indicator: Hessian for 2nd scheme, higher order derivatives for higher order schemes

Anisotropic refinement: Laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ **flow around the NACA0012 airfoil**



DG(2), i.e. 3rd order, with adjoint-based refinement, error measured in c_{dp}

Comparison of:

- **isotropic refinement**
- **anisotropic jump indicator**
- **anisotropic derivative indicator (3rd derivatives)**

Adjoint consistency: Motivation

The interior penalty (IP) discontinuous Galerkin discretization of Poisson's equation [Harriman, Houston, Senior, Süli]

- H^1: **SIPG/NIPG:** h^p
- L^2: **SIPG:** h^{p+1}; **NIPG:** h^p **(theory),** h^{p+1} **for** p **odd (numerics)**
- $J(\cdot)$: **SIPG:** h^{2p}; **NIPG:** h^p **(theory),** h^{p+1} **for** p **odd (numerics)**

In case of SIPG and a target functional $J(u) = \int_{\Gamma_D} \mathbf{n} \cdot \nabla u j_D \,\mathrm{d}s$ **the modification**

$$\widehat{J}(u_h) = J(u_h) - \int_{\Gamma_D} u_h \delta j_D \,\mathrm{d}s \tag{1}$$

is required for recovering adjoint-consistency [Harriman, Gavaghan, Süli].

For the compressible Euler equation [Lu, Darmofal] proposed to use a modified discretization of boundary conditions and of the target functional for recovering an adjoint-consistent DG discretization.

Consistency and adjoint consistency: Linear problems

Continuous primal problem: $Lu = f$ **in** Ω, $Bu = g$ **on** Γ, (2)

Target functional: $J(u) = \int_\Omega j_\Omega\, u \,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma\, u \,\mathrm{d}s,$

Continuous adjoint problem: $L^* z = j_\Omega$ **in** Ω, $B^* z = j_\Gamma$ **on** Γ. (3)

Discrete primal problem: **Find** u_h **such that** $\mathcal{B}(u_h, v) = l(v) \quad \forall v \in V_h,$

Consistency: **For** u **solution to** (2) $\mathcal{B}(u, v) = l(v) \quad \forall v \in V,$

Discrete adjoint problem: **Find** z_h **such that** $\mathcal{B}(w, z_h) = J(w) \quad \forall w \in V_h,$

Adjoint consistency: **For** z **solution to** (3) $\mathcal{B}(w, z) = J(w) \quad \forall w \in V.$

Consistency and adjoint consistency: Nonlinear problems

Continuous primal problem: $Nu = 0$ **in** Ω, $Bu = 0$ **on** Γ, (4)

Target functional: $$J(u) = \int_\Omega j_\Omega(u)\,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma(u)\,\mathrm{d}s,$$

with linearization: $$J'[u](w) = \int_\Omega j'_\Omega[u]\,w\,\mathrm{d}\mathbf{x} + \int_\Gamma j'_\Gamma[u]\,w\,\mathrm{d}s,$$

Continuous adjoint problem: $(N'[u])^* z = j'_\Omega[u]$ **in** Ω, $(B'[u])^* z = j'_\Gamma[u]$ **on** Γ, (5)

Discrete primal problem: **Find** u_h **such that** $\mathcal{N}(u_h, v) = 0 \quad \forall v \in V_h$,

Consistency: **For** u **solution to** (4) $\mathcal{N}(u, v) = 0 \quad \forall v \in V$,

Discrete adjoint problem: **Find** z_h **such that** $\mathcal{N}'[u](w, z_h) = J'[u](w) \quad \forall w \in V_h$,

Adjoint consistency: **For** z **solution to** (5) $\mathcal{N}'[u](w, z) = J'[u](w) \quad \forall w \in V$.

Consistency

Rewrite the discrete primal problem: Find u_h such that

$$\mathcal{N}(u_h, v) = 0 \quad \forall v \in V_h,$$

as follows: Find u_h such that

$$\sum_{\kappa\in\mathcal{T}_h} \int_\kappa R(u_h) v \, \mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} r(u_h) v \, \mathrm{d}s + \int_\Gamma r_\Gamma(u_h) v \, \mathrm{d}s = 0 \quad \forall v \in V_h.$$

Then, the discretization is consistent provided

$$R(u) = 0 \ \text{ in } \kappa, \kappa \in \mathcal{T}_h, \qquad r(u) = 0 \ \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \qquad r_\Gamma(u) = 0 \ \text{ on } \Gamma,$$

holds for the exact primal solution u.

Adjoint consistency

Rewrite the discrete adjoint problem: Find z_h such that

$$\mathcal{N}'[u](w, z_h) = J'[u](w) \quad \forall w \in V_h,$$

as follows: Find u_h such that

$$\sum_{\kappa\in\mathcal{T}_h} \int_\kappa w\, R^*(z_h)\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} w\, r^*(z_h)\,\mathrm{d}s + \int_\Gamma w\, r_\Gamma^*(z_h)\,\mathrm{d}s = 0 \quad \forall w \in V.$$

Then, the discretization is adjoint consistent provided

$$R^*(z) = 0 \ \text{ in } \kappa, \kappa \in \mathcal{T}_h, \qquad r^*(z) = 0 \ \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \qquad r_\Gamma^*(z) = 0 \ \text{ on } \Gamma.$$

holds for the (continuous) adjoint solution z.

Consistent modification of target functionals

Definition: $\tilde{J}(u)$ **is a consistent modification of** $J(u)$
if $\tilde{J}(u) = J(u)$ **holds for the exact solution** u**.**

We replace $J(u)$ **by** $\tilde{J}(u)$ **with**

$$\tilde{J}(u) = J(i(u)) + \int_{\Gamma} r_J(u)\, \mathrm{d}s.$$

This modification is consistent if $i(u) = u$ **and** $r_J(u) = 0$**.**

Derivation of consistent/adjoint consistency discretizations

Given a baseline (and possibly non adjoint consistent) discretization

- **Derive the primal residuals $R(u_h)$, $r(u_h)$ and $r_\Gamma(u_h)$.**
- **Derive adjoint residuals $R^*(z_h)$, $r^*(z_h)$ and $r_\Gamma^*(z_h)$.**
- **Modify the discretization of element and interior terms, of boundary terms and of the target functional such that**

$$R(u) = 0 \ \text{ in } \kappa, \kappa \in \mathcal{T}_h, \qquad r(u) = 0 \ \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \qquad r_\Gamma(u) = 0 \ \text{ on } \Gamma,$$

and

$$R^*(z) = 0 \ \text{ in } \kappa, \kappa \in \mathcal{T}_h, \qquad r^*(z) = 0 \ \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \qquad r_\Gamma^*(z) = 0 \ \text{ on } \Gamma,$$

holds for the exact primal and adjoint solutions u and z.

Then, we have a consistent and adjoint consistent discretization.

Outcome of adjoint consistency analysis

SIPG for Poisson's equation with

$$J(u) = \int_{\Gamma_D} \mathbf{n} \cdot \nabla u j_D \, \mathrm{d}s.$$

Use following modification of $J(\cdot)$ (we call it *IP modification* of $J(\cdot)$)

$$\tilde{J}(u_h) = J(u_h) + \int_{\Gamma_D} r_J(u_h) \, \mathrm{d}s, \text{ with } r_J(u_h) = \delta(u_h - g_D) z_\Gamma, \text{ and } z_\Gamma = -j_D,$$

which is consistent as $r_J(u) = 0$.

We recover the modification [Harriman, Houston, Senior, Süli] through linearization

$$\tilde{J}[u](w) = J(w) - \int_{\Gamma_D} \delta w \, j_D \, \mathrm{d}s$$

Outcome of adjoint consistency analysis

DG for the compressible Euler equations with (pressure-induced) force coefficient

$$J(\mathbf{u}_h) = \frac{1}{C_\infty} \int_{\Gamma_W} p\, \mathbf{n} \cdot \psi \, \mathrm{d}s.$$

Instead of

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) \quad \textbf{and} \quad J(\mathbf{u}_h)$$

use

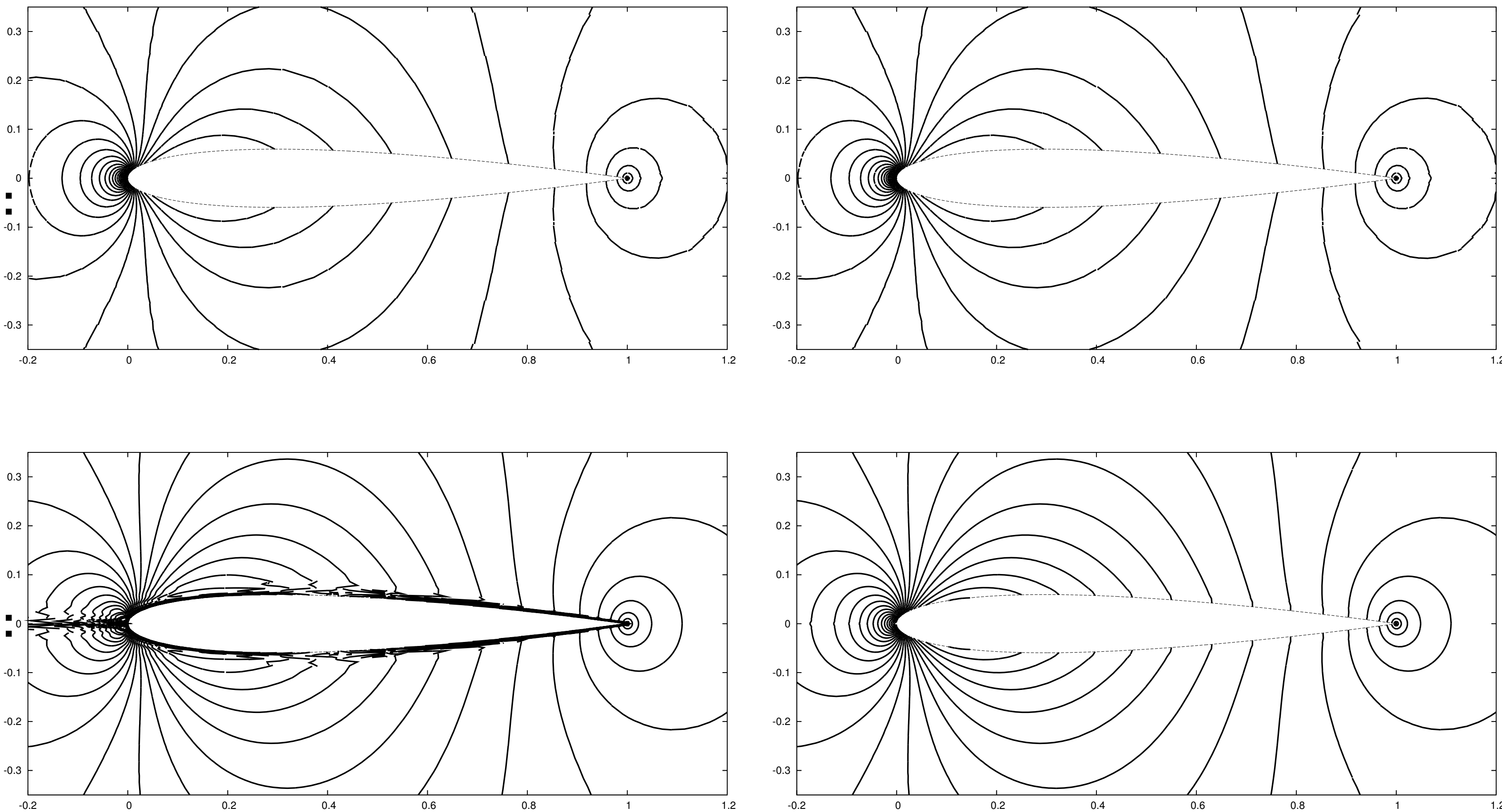
$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{n} \quad \textbf{and} \quad \tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h)).$$

$\tilde{J}(\mathbf{u}_h)$ **is consistent as** $\mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{u}$**.**

Numerical comparison: standard DG vs. adjoint consistent DG

Primal solution: Mach isolines

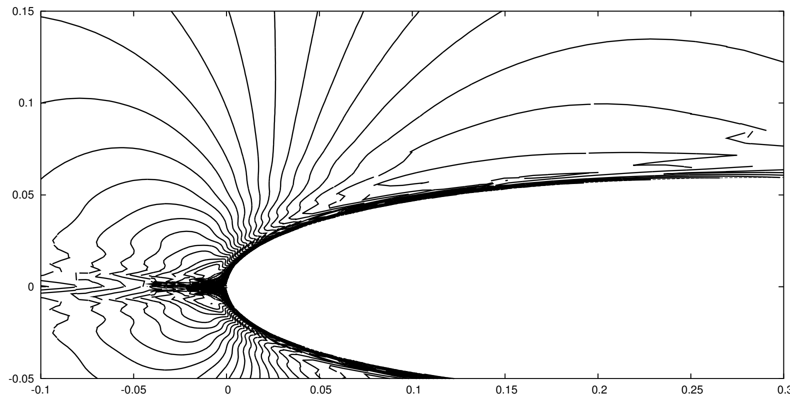
Discrete adjoint solution: z_1 isolines



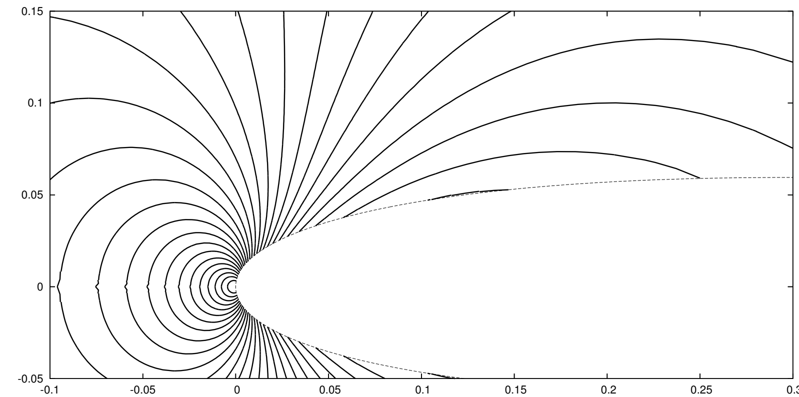
standard DG **adjoint consistent DG**

Numerical comparison: standard DG vs. adjoint consistent DG

Discrete adjoint solution: z_1 isolines



standard DG



adjoint consistent DG

primal solution: error in cdp

# elem.	standard DG	adj.cons. DG
768	-5.008e-03	-3.800e-03
1242	-1.783e-03	-8.833e-04
2061	-5.422e-04	-2.302e-04
3339	-1.617e-04	-8.405e-05
5535	-5.060e-05	-3.754e-05

Gain in accuracy: a factor of about 2

Outcome of adjoint consistency analysis

SIPG for the compressible Navier-Stokes equations with (total) force coefficient

$$J(\mathbf{u}_h) = \frac{1}{C_\infty} \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \psi \,\mathrm{d}s.$$

Instead of

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}), \;\; G_\Gamma(\mathbf{u}_h^+) = G(\mathbf{u}_h^+), \qquad \textbf{and} \quad J(\mathbf{u}_h),$$

use

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{n}, \qquad G_\Gamma(\mathbf{u}_h^+) = G(\mathbf{u}_\Gamma(\mathbf{u}_h^+)), \;\; \textbf{and} \quad \tilde{J}(\mathbf{u}_h),$$

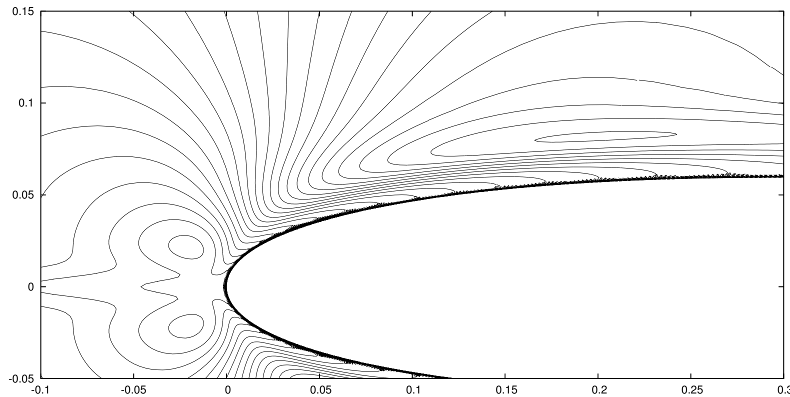
with

$$\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h)) + \int_{\Gamma_W} \delta\left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right) \cdot \mathbf{z}_\Gamma \,\mathrm{d}s,$$

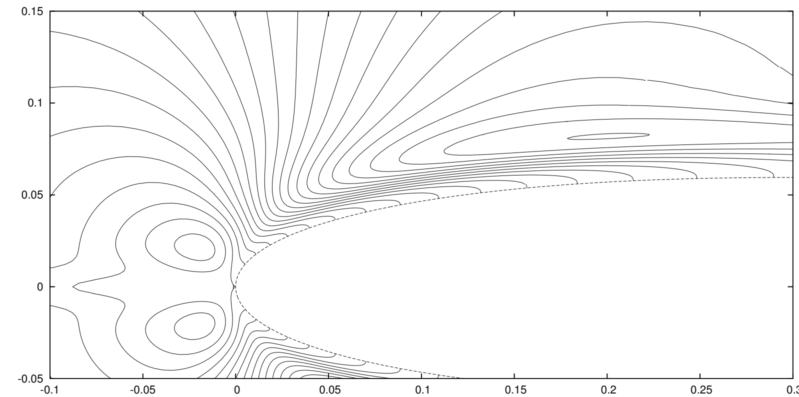
and $\mathbf{z}_\Gamma = \frac{1}{C_\infty}(0, \psi_1, \psi_2, 0)^\top$**.**

Numerical comparison: standard DG vs. adjoint consistent DG

Discrete adjoint solution: z_4 isolines



standard DG



adjoint consistent DG

primal solution: error in cd

# elem.	standard DG		adj.cons. DG		ac. DG/IP mod.	
3072	-3.164e-03	-	1.502e-03	-	-1.243e-03	-
12288	8.048e-04	3.9	3.682e-05	40.8	6.994e-04	1.8
49152	4.519e-04	1.8	-1.139e-06	32.3	4.795e-04	1.5

Gain in accuracy: a factor of about 2-400!!

Conclusion

Part I: Higher order discontinuous Galerkin discretization of the compressible Euler and Navier-Stokes equations

- Same accuracy on coarser meshes & less computational time than for 2nd order
- Accurate error estimation with respect to target quantities
- Efficient adjoint-based (goal-oriented) adaptive mesh refinement

Part II: Adjoint consistent Discontinuous Galerkin discretizations

- For an adjoint consistent discretization the discrete adjoint problem is a consistent discretization of the continuous adjoint problem
- We have introduced a framework for analyzing adjoint consistency including the definition of consistent modifications of target functionals
- Analysis of DG (Poisson's equ., compressible Euler & Navier-Stokes equations)
- Numerics: accuracy (and order of convergence) is significantly improved

Outlook: Next steps towards application in industry

- Extension of the flow solver
  - to three-dimensional turbulent, high Reynolds flow
- The above extension also for
  - the computation of the adjoint solution
  - the evaluation of refinement indicators (residual-based and adjoint-based)
  - and the error estimation with respect to aerodynamical force coefficients
- Error estimation and adaptivity with respect to multiple target quantities
- Extension to hybrid meshes
- Extension to hp-refinement
- Efficient solution algorithms
  - linear and nonlinear multigrid
  - h- and p- multigrid

EU Project: ADIGMA

Adaptive Higher-Order Variational Methods for Aerodynamic Applications in Industry

Start was 1st of Sept 2006

Co-ordinator:	DLR
Industrial partners:	Airbus-D, Airbus-F, Dassault, Alenia, EADS-M
Research institutes:	DLR, ONERA, NLR, FOI, INRIA, VKI
Universities:	Uni Bergamo, Uni Twente, Uni Swansea, Uni Nottingham
	Uni Stuttgart, Uni Warsaw, Uni Prague, ENSAM, (Uni Nanjing)
SMEs:	ARA, CENAERO

Topics: Mainly Discontinuous Galerkin methods, also Residual Distribution Schemes
Multigrid, Newton-like methods, adaptation, error estimation, hp-refinement